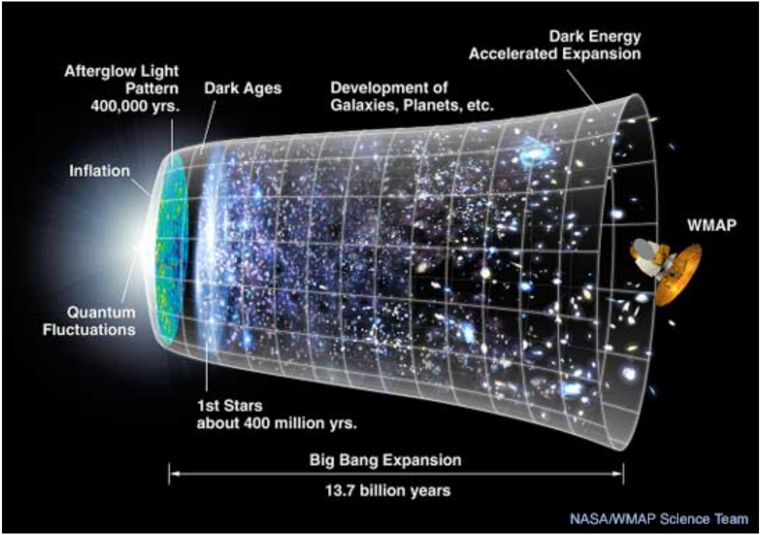
Lectures on modern cosmology

Alexei A. Starobinsky

Landau Institute for Theoretical Physics RAS

Faculty of Physics, National Research University
"Higher School of Economics"

Moscow, May 2020


Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years
NASA/WMAP Science Team

Four epochs of the history of the Universe

$H \equiv \frac{\dot{a}}{a}$ where $a(t)$ is a scale factor of an isotropic homogeneous spatially flat universe (a Friedmann-Lemaître-Robertson-Walker background):

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) + small\ perturbations$$

The history of the Universe in one line: four main epochs

$$? \longrightarrow DS \Longrightarrow FLRWRD \Longrightarrow FLRWMD \Longrightarrow \overline{DS} \longrightarrow ?$$

Geometry

$$|\dot{H}| << H^2 \Longrightarrow H = \frac{1}{2t} \Longrightarrow H = \frac{2}{3t} \Longrightarrow |\dot{H}| << H^2$$

Physics

$$p \approx -\rho \Longrightarrow p = \rho/3 \Longrightarrow p \ll \rho \Longrightarrow p \approx -\rho$$

Duration in terms of the number of e-folds $\ln(a_{fin}/a_{in})$

$$> 60 \qquad \sim 55 \qquad 7.5 \qquad 0.5$$

Principal epochs of the Universe evolution – before 1979

The history of the Universe in one line: two principal epochs

$$? \longrightarrow FLRWRD \Longrightarrow FLRWMD \longrightarrow ?$$

Geometry

$$H = \frac{1}{2t} \Longrightarrow H = \frac{2}{3t}$$

Physics

$$p = \rho/3 \Longrightarrow p \ll \rho$$

Present matter content of the Universe

In terms of the critical density

$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 0.920 \times 10^{-29} \left(\frac{H_0}{70}\right)^2 \mathrm{g/cm^3}$

$\Omega_i = \frac{\rho_i}{\rho_{crit}}, \quad \sum_i \Omega_i = 1$

where the Hubble constant $H_0 = 70 \pm 3$ km/s/Mpc

(neglecting spatial curvature - less than 0.5%):

- Baryons (p,n) and leptons (e^-) $\approx 5\%$
  No primordial antimatter.
- Photons (γ) 5.0×10^{-5}
  $T_\gamma = (2.72548 \pm 0.00057)\mathrm{K}$
- Neutrinos ($\nu_e, \ \nu_\mu, \ \nu_\tau$) $< 0.5\%$

$$\sum_i m_{\nu i} < 0.2 \ \mathrm{eV}, \quad \sum_i m_{\nu i} = 46\Omega_\nu \left(\frac{H_0}{70}\right) \mathrm{eV}.$$

- Non-relativistic non-baryonic dark matter $\approx 25\%$
- Dark energy $\approx 70\%$

Dark matter

Dark matter and dark energy are seen through gravitational interaction only – we know the structure of their effective energy-momentum tensor.

DM - non-relativistic, gravitationally clustered.
DE - relativistic, unclustered.
Definition of their effective EMT – through equations (conventional).
DM - through the generalized Poisson equation:

$$\frac{\triangle\Phi}{a^2} = 4\pi G(\rho - \rho_0(t)).$$

$\Phi(\mathbf{r}, t)$ is measured using the motion of 'test particles' in it.
a) Stars in galaxies $\rightarrow$ rotation curves.
b) Galaxies $\rightarrow$ peculiar velocities.
c) Hot gas in galaxies $\rightarrow$ X-ray profiles.
d) Photons $\rightarrow$ gravitational lensing (strong and weak).

Observations: DM is non-relativistic, has a dust-like EMT – $p \ll \epsilon = \rho c^2, \quad p > 0$, collisionless in the first approximation – $\sigma/m < 1~\mathrm{cm}^2/\mathrm{g}$, and has the same spatial distribution as visible matter for scales exceeding a few Mpc.

Ground experiments: very weakly interacting with baryonic matter, $\sigma < 10^{-43}~\mathrm{cm}^2$ for $m \sim (50 - 100)$ GeV.

Dark energy

Two cases where DE shows itself:
1) inflation in the early Universe – primordial DE,
2) present accelerated expansion of the Universe – present DE.

Quantitative and internally self-consistent definition of its effective EMT - through gravitational field equations conventionally written in the Einstein form:

$$\frac{1}{8\pi G}\left(R_{\mu}^{\nu}-\frac{1}{2}\,\delta_{\mu}^{\nu}R\right)=\left(T_{\mu\,(vis)}^{\nu}+T_{\mu\,(DM)}^{\nu}+T_{\mu\,(DE)}^{\nu}\right)\ ,$$

$G=G_0=const$ - the Newton gravitational constant measured in laboratory.
In the absence of direct interaction between DM and DE:

$$T_{\mu\,(DE);\nu}^{\nu}=0\ .$$

Possible forms of DE

- Physical DE.
  New non-gravitational field of matter. DE proper place – in the rhs of gravity equations.
- Geometrical DE.
  Modified gravity. DE proper place – in the lhs of gravity equations.
- Λ - intermediate case.

Observations: $T^{\nu}_{\mu\,(DE)}$ is very close to $\Lambda\delta^{\nu}_{\mu}$ for the concrete solution describing our Universe;

$$< w_{DE} >= -1.03 \pm 0.03$$

where $w_{DE} \equiv p_{DE}/\epsilon_{DE}$.
$w_{DE} > -1$ – normal case,
$w_{DE} < -1$ – phantom case,
$w_{DE} \equiv -1$ – the exact cosmological constant ("vacuum energy").

Four fundamental cosmological constants

One-to-one relation to the four epochs of the history of the Universe.
A fundamental theory beyond each of these constants.

- Characteristic amplitude of primordial scalar (adiabatic) perturbations.

$$< \mathcal{R}^2(\mathbf{r}) >= \int \frac{P_{\mathcal{R}}(k)}{k}\, dk, \;\; P_{\mathcal{R}}(k) = 2.10\times 10^{-9} \left(\frac{k}{k_0}\right)^{n_s-1}$$

$$k_0 = 0.05\mathrm{Mpc}^{-1}, \quad n_s - 1 = -0.035 \pm 0.004$$

Theory of initial conditions – inflation. Its simplest model (1980) predicted the slope of the spectrum relating it finally to $N_H = \ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$:

$$n_s - 1 = -\frac{2}{N}$$

where $N = N_H - \mathcal{O}(10)$ is the number of e-folds from the end of inflation.

- Baryon to photon ratio.

$$\frac{n_b}{n_\gamma} = 6.01 \times 10^{-10} \, \frac{\Omega_b h^2}{0.0022} \, \left(\frac{2.725}{T_\gamma(\mathrm{K})}\right)^3 , h = \frac{H_0}{100} \, .$$

Theory of baryogenesis.

- Baryon to total non-relativistic matter density.

$$\frac{\rho_b}{\rho_m} = 0.167 \, \frac{\Omega_b}{0.05} \, \frac{0.3}{\Omega_m} \, .$$

Theory of dark matter.

- Energy density of present dark energy.

$$\rho_{DE} = \frac{\epsilon_{DE}}{c^2} = 6.44 \times 10^{-30} \, \frac{\Omega_{DE}}{0.7} \, \left(\frac{H_0}{70}\right)^2 \, \mathrm{g/cm^3} \, ,$$

$$\frac{G^2 \hbar \epsilon_{DE}}{c^7} = 1.25 \times 10^{-123} \, \frac{\Omega_{DE}}{0.7} \, \left(\frac{H_0}{70}\right)^2 \, .$$

Theory of present dark energy.

Necessary condition for galaxy formation

$$\sqrt{P_{\mathcal{R}}}\left(\frac{t_{\Lambda}}{t_{eq}}\right)^{2/3} \gtrsim 1$$

It is also necessary for stars, planets and life appearance. Thus, the four fundamental cosmological constants

$$A_1 = 2.1\times10^{-9},\ A_2 = 6.01\times10^{-10},\ A_3 = 0.167,\ A_4 = 1.25\times10^{-123}$$

should satisfy the inequality

$$\left(\frac{m_p}{M_{Pl}}\right)^4\left(\frac{A_2}{A_3}\right)^4\frac{A_1^{3/2}}{A_4} \gtrsim 1$$

In fact, the left-hand side is equal to 0.46, so it is satisfied but "just so".

Inflation

The inflationary scenario is based on the two cornerstone independent ideas (hypothesis):

1. Existence of inflation (or, quasi-de Sitter stage) – a stage of accelerated, close to exponential expansion of our Universe in the past preceding the hot Big Bang with decelerated, power-law expansion.

2. The origin of all inhomogeneities in the present Universe is the effect of gravitational creation of particles and field fluctuations during inflation from the adiabatic vacuum (no-particle) state for Fourier modes covering all observable range of scales (and possibly somewhat beyond).

Existing analogies in other areas of physics.
1. The present dark energy.
2. Creation of electrons and positrons in an external elecric field.

Outcome of inflation

In the super-Hubble regime ($k \ll aH$) in the coordinate representation:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dx^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\mathcal{R}(\mathbf{r})\delta_{lm} + \sum_{a=1}^{2} g^{(a)}(\mathbf{r})\, e_{lm}^{(a)}$$

$$e_l^{l(a)} = 0, \;\; g^{(a)}_{,l}\, e_m^{l(a)} = 0, \;\; e_{lm}^{(a)}\, e^{lm(a)} = 1$$

$\mathcal{R}$ describes primordial scalar perturbations, g – primordial tensor perturbations (primordial gravitational waves (GW)).

The most important quantities:

$$n_s(k) - 1 \equiv \frac{d \ln P_{\mathcal{R}}(k)}{d \ln k}, \quad r(k) \equiv \frac{P_g}{P_{\mathcal{R}}}$$

In fact, metric perturbations h_{lm} are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in $\mathcal{R}, g$).

In particular:

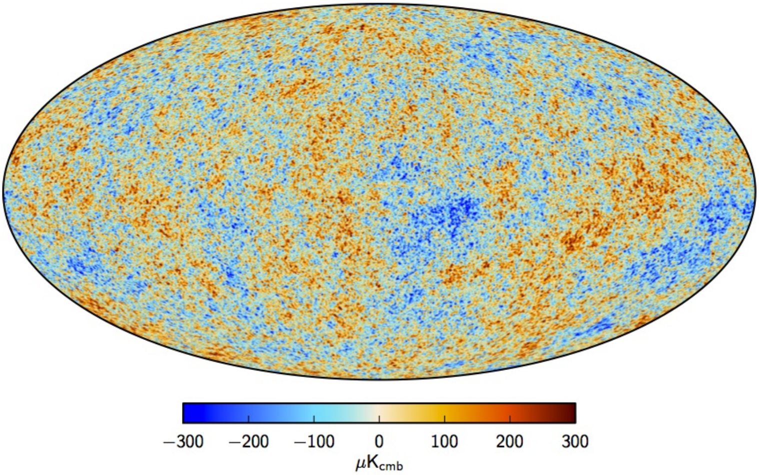
$$\hat{\mathcal{R}}_k = \mathcal{R}_k \, i(\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}) + \mathcal{O}\left((\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}})^2\right) + ... + \mathcal{O}(10^{-100})(\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}) + ,,,$$

The last term is time dependent, it is affected by physical decoherence and may become larger, but not as large as the second term.

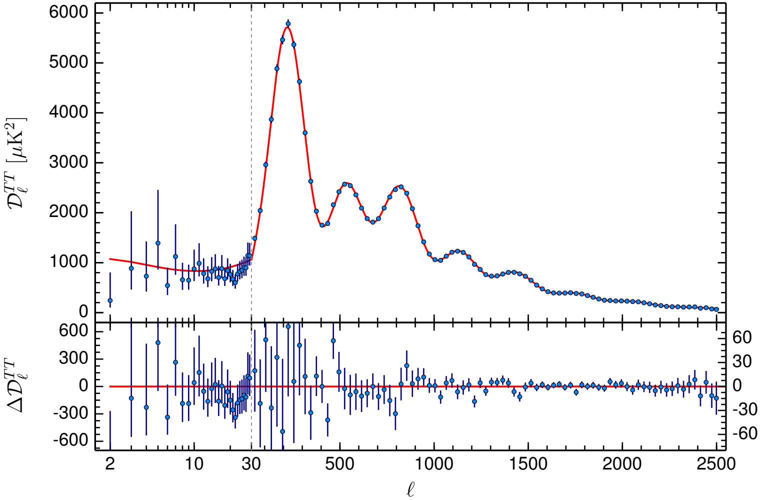
Remaining quantum coherence: deterministic correlation between $\mathbf{k}$ and $-\mathbf{k}$ modes - shows itself in the appearance of acoustic oscillations.

CMB temperature anisotropy

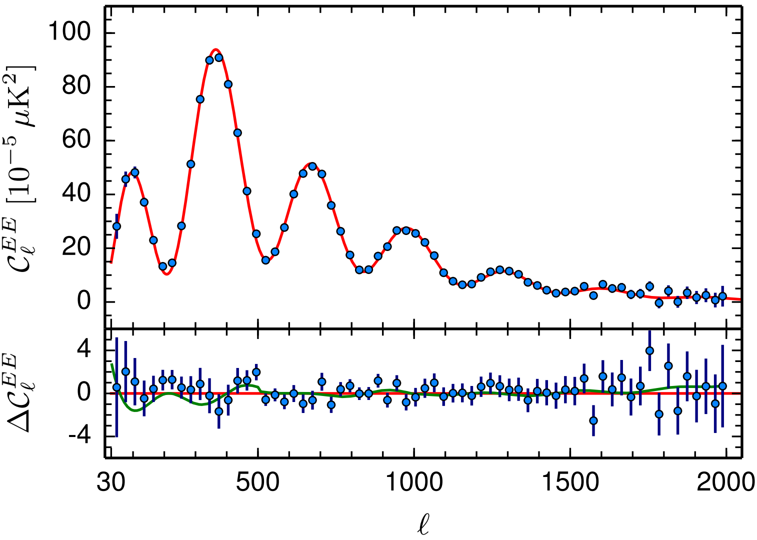
Planck-2015: P. A. R. Ade et al., arXiv:1502.01589



CMB temperature anisotropy multipoles



CMB E-mode polarization multipoles



Two observational parameters of inflation

Now we have numbers: N. Agranim et al., arXiv:1807.06209

The primordial spectrum of scalar perturbations has been measured and its deviation from the flat spectrum $n_s = 1$ in the first order in $|n_s - 1| \sim N_H^{-1}$ has been discovered (using the multipole range $\ell > 40$):

$$< \mathcal{R}^2(\mathbf{r}) >= \int \frac{P_{\mathcal{R}}(k)}{k}\, dk, \;\; P_{\mathcal{R}}(k) = (2.10 \pm 0.03)\cdot 10^{-9} \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$$k_0 = 0.05 \ \mathrm{Mpc}^{-1}, \quad n_s - 1 = -0.035 \pm 0.004$$

Two fundamental observational constants of cosmology in addition to the three known ones (baryon-to-photon ratio, baryon-to-matter density and the cosmological constant). Existing inflationary models can predict (and predicted, in fact) one of them, namely $n_s - 1$, relating it finally to $N_H = \ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$. (note that $(1 - n_s)N_H \sim 2$).

Physical scales related to inflation

"Naive" estimate where I use the reduced Planck mass $\tilde{M}_{Pl} = (8\pi G)^{-1}$.

I. Curvature scale

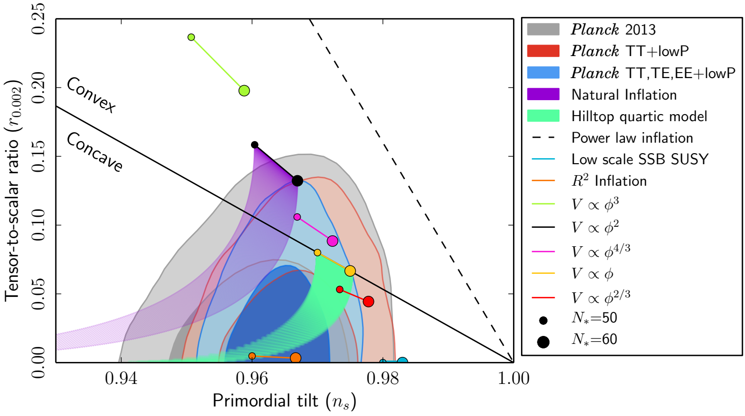
$$H \sim \sqrt{P_{\mathcal{R}}}\tilde{M}_{Pl} \sim 10^{14}\mathrm{GeV}$$

II. Inflaton mass scale

$$|m_{infl}| \sim H\sqrt{|1 - n_s|} \sim 10^{13}\mathrm{GeV}$$

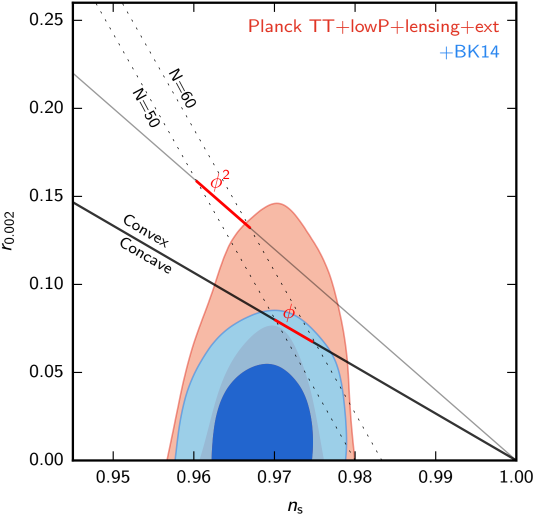
New range of mass scales significantly less than the GUT scale.

Direct approach: comparison with simple smooth models



Combined BICEP2/Keck Array/Planck results

P. A. R. Ade et al., Phys. Rev. Lett. 116, 031302 (2016)



The simplest models producing the observed scalar slope

1. The $R + R^2$ model (Starobinsky, 1980):

$$\mathcal{L} = \frac{f(R)}{16\pi G}, \quad f(R) = R + \frac{R^2}{6M^2}$$

$$M = 2.6 \times 10^{-6} \left(\frac{55}{N}\right) M_{Pl} \approx 3.1 \times 10^{13}\,\mathrm{GeV}$$

$$n_s - 1 = -\frac{2}{N} \approx -0.036, \quad r = \frac{12}{N^2} \approx 0.004$$

$$N = \ln \frac{k_f}{k} = \ln \frac{T_\gamma}{k} - \mathcal{O}(10), \quad H_{dS}(N = 55) = 1.4 \times 10^{14}\,\mathrm{GeV}$$

2. The same prediction from a scalar field model with $V(\phi) = \frac{\lambda\phi^4}{4}$ at large ϕ and strong non-minimal coupling to gravity $\xi R\phi^2$ with $\xi < 0, \ |\xi| \gg 1$, including the Higgs inflationary model (Bezrukov and Shaposhnikov, 2008).

The simplest purely geometrical inflationary model

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{N^2}{288\pi^2 P_{\mathcal{R}}(k)} R^2 + (\text{small rad. corr.})$$

$$= \frac{R}{16\pi G} + 5.1 \times 10^8 \, R^2 + (\text{small rad. corr.})$$

The quantum effect of creation of particles and field fluctuations works twice in this model:
a) at super-Hubble scales during inflation, to generate space-time metric fluctuations;
b) at small scales after inflation, to provide scalaron decay into pairs of matter particles and antiparticles (AS, 1980, 1981).

Weak dependence of the time t_r when the radiation dominated stage begins:

$$N(k) \approx N_H + \ln \frac{a_0 H_0}{k} - \frac{1}{3} \ln \frac{M_{\rm Pl}}{M} - \frac{1}{6} \ln(M_{\rm Pl} t_r)$$

Evolution of the $R + R^2$ model

1. During inflation ($H \gg M$):

$$H = \frac{M^2}{6}(t_f - t), \quad |\dot{H}| \ll H^2$$

.

2. After inflation ($H \ll M$):

$$a(t) \propto t^{2/3}\left(1 + \frac{2}{3Mt}\sin M(t - t_1)\right)$$

The most effective decay channel: into minimally coupled scalars with $m \ll M$. Then the formula

$$\frac{1}{\sqrt{-g}}\frac{d}{dt}(\sqrt{-g}\,n_s) = \frac{R^2}{576\pi}$$

(Ya. B. Zeldovich and A. A. Starobinsky, JETP Lett. 26, 252 (1977)) can be used for simplicity, but the full integral-differential system of equations for the Bogoliubov α_k, β_k coefficients and the average EMT was in fact solved in AS (1981). Scalaron decay into graviton pairs is suppressed (A. A. Starobinsky, JETP Lett. 34, 438 (1981)).

For this channel of the scalaron decay:

$$N(k) \approx N_H + \ln\frac{a_0 H_0}{k} - \frac{5}{6}\ln\frac{M_{\rm Pl}}{M}$$

Possible microscopic origins of this phenomenological model.

1. Follow the purely geometrical approach and consider it as the specific case of the fourth order gravity in 4D

$$\mathcal{L} = \frac{R}{16\pi G} + AR^2 + BC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + (\text{small rad. corr.})$$

for which $A \gg 1, \ A \gg |B|$. Approximate scale (dilaton) invariance and absence of ghosts in the curvature regime $A^{-2} \ll (RR)/M_P^4 \ll B^{-2}$.

One-loop quantum-gravitational corrections are small (their imaginary parts are just the predicted spectra of scalar and tensor perturbations), non-local and qualitatively have the same structure modulo logarithmic dependence on curvature.

2. Another, completely different way:

consider the $R + R^2$ model as an approximate description of GR + a non-minimally coupled scalar field with a large negative coupling ξ ($\xi_{conf} = \frac{1}{6}$) in the gravity sector::

$$\mathcal{L} = \frac{R}{16\pi G} - \frac{\xi R \phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \; .$$

Geometrization of the scalar:

for a generic family of solutions during inflation and even for some period of non-linear scalar field oscillations after it, the scalar kinetic term can be neglected, so

$$\xi R \phi = -V'(\phi) + \mathcal{O}(|\xi|^{-1}) \; .$$

No conformal transformation, we remain in the the physical (Jordan) frame!

These solutions are the same as for $f(R)$ gravity with

$$\mathcal{L} = \frac{f(R)}{16\pi G}, \ f(R) = R - \frac{\xi R\phi^2(R)}{2} - V(\phi(R)).$$

For $V(\phi) = \frac{\lambda(\phi^2-\phi_0^2)^2}{4}$, this just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi|R/\lambda$.

The same theorem is valid for a multi-component scalar field.

More generally, R^2 inflation (with an arbitrary n_s, r) serves as an intermediate dynamical attractor for a large class of scalar-tensor gravity models.

Inflation in the mixed Higgs-R^2 Model

M. He, A. A. Starobinsky and J. Yokoyama, JCAP **1805**, 064 (2018).

$$\mathcal{L} = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right) - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - \frac{\lambda\phi^4}{4}, \;\; \xi < 0, \;\; |\xi| \gg 1$$

In the attractor regime during inflation (and even for some period after it), we return to the $f(R) = R + \frac{R^2}{6M^2}$ model with the renormalized scalaron mass $M \to \tilde{M}$:

$$\frac{1}{\tilde{M}^2} = \frac{1}{M^2} + \frac{24\pi\xi^2 G}{\lambda}$$

Inflation in GR

Inflation in GR with a minimally coupled scalar field with some potential.

In the absence of spatial curvature and other matter:

$$H^2 = \frac{\kappa^2}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

where $\kappa^2 = 8\pi G$ ($\hbar = c = 1$).

Reduction to the first order equation

It can be reduced to the first order Hamilton-Jacobi-like equation for $H(\phi)$. From the equation for $\dot{H}$, $\frac{dH}{d\phi} = -\frac{\kappa^2}{2}\dot{\phi}$. Inserting this into the equation for H^2, we get

$$\frac{2}{3\kappa^2}\left(\frac{dH}{d\phi}\right)^2 = H^2 - \frac{\kappa^2}{3}V(\phi)$$

Time dependence is determined using the relation

$$t = -\frac{\kappa^2}{2}\int \left(\frac{dH}{d\phi}\right)^{-1} d\phi$$

However, during oscillations of ϕ, $H(\phi)$ acquires non-analytic behaviour of the type $const + \mathcal{O}(|\phi - \phi_1|^{3/2})$ at the points where $\dot{\phi} = 0$, and then the correct matching with another solution is needed.

Inflationary slow-roll dynamics

Slow-roll occurs if: $|\ddot{\phi}| \ll H|\dot{\phi}|$, $\dot{\phi}^2 \ll V$, and then $|\dot{H}| \ll H^2$.

Necessary conditions: $|V'| \ll \kappa V$, $|V''| \ll \kappa^2 V$. Then

$$H^2 \approx \frac{\kappa^2 V}{3}, \quad \dot{\phi} \approx -\frac{V'}{3H}, \quad N \equiv \ln \frac{a_f}{a} \approx \kappa^2 \int_{\phi_f}^{\phi} \frac{V}{V'}\, d\phi$$

First obtained in A. A. Starobinsky, Sov. Astron. Lett. 4, 82 (1978) in the $V = \frac{m^2\phi^2}{2}$ case and for a bouncing model.

Quantum generation of perturbations during inflation

Quantization with the adiabatic vacuum initial condition (in the tensor case, omitting the polarization tensor):

$$\hat{\phi} = (2\pi)^{-3/2} \int \left[\hat{a}_{\mathbf{k}}\, \phi_k(\eta)\, e^{-i\mathbf{kr}} + \hat{a}^{\dagger}_{\mathbf{k}}\, \phi^*_k\, e^{i\mathbf{kr}} \right] d^3k$$

where ϕ stands for ζ, g^a correspondingly and ϕ_k satisfies the equation

$$\frac{1}{f}(f\phi_k)'' + \left(k^2 - \frac{f''}{f} \right) \phi_k = 0, \quad \eta = \int \frac{dt}{a(t)}$$

For GW: $f = a$, for scalar perturbations in scalar field driven inflation in GR: $f = \frac{a\dot{\phi}}{H}$ where, in turn, the background scalar field satisfies the equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

How the two basic hypothesis of the inflationary paradigm work.

I. Inflationary background: $t = \infty$ corresponds to $\eta = 0$ and $H(\eta) \equiv \frac{a'}{a^2}$ is bounded and slowly decreasing in this limit, so that $\frac{f''}{f} \sim \frac{2}{\eta^2}$. Then

$$\eta \to -0 : \quad \phi_k(\eta) \to \phi(k) = const, \quad P(k) = \frac{k^3|\phi^2(k)|}{2\pi^2}$$

II. Adiabatic vacuum initial condition:

$$\eta \to -\infty : \quad \phi_k(\eta) = \frac{e^{-ik\eta}}{f\sqrt{2k}}$$

Combining both conditions:

$$\phi_k(\eta) \approx \frac{e^{-ik\eta}}{f\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right), \; a(\eta) \approx \frac{1}{H(\eta)|\eta|}$$

Spectral predictions of the one-field inflationary scenario in GR

Scalar (adiabatic) perturbations:

$$P_\zeta(k) = \frac{H_k^4}{4\pi^2\dot{\phi}^2} = \frac{GH_k^4}{\pi|\dot{H}|_k} = \frac{128\pi G^3 V_k^3}{3V_k'^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$. Through this relation, the number of e-folds from the end of inflation back in time $N(t)$ transforms to $N(k) = \ln\frac{k_f}{k}$ where $k_f = a(t_f)H(t_f)$, t_f denotes the end of inflation.

The spectral slope

$$n_s(k) - 1 \equiv \frac{d\ln P_\zeta(k)}{d\ln k} = \frac{1}{\kappa^2}\left(2\frac{V_k''}{V_k} - 3\left(\frac{V_k'}{V_k}\right)^2\right)$$

is small by modulus – confirmed by observations!

Tensor perturbations (A. A. Starobinsky, JETP Lett. 50, 844 (1979)):

$$P_g(k) = \frac{16GH_k^2}{\pi}; \quad n_g(k) \equiv \frac{d\ln P_g(k)}{d\ln k} = -\frac{1}{\kappa^2}\left(\frac{V'_k}{V_k}\right)^2$$

The consistency relation:

$$r(k) \equiv \frac{P_g}{P_\zeta} = \frac{16|\dot{H}_k|}{H_k^2} = 8|n_g(k)|$$

Tensor perturbations are always suppressed by at least the factor $\sim 8/N(k)$ compared to scalar ones. For the present Hubble scale, $N(k_H) = (50-60)$. Typically, $|n_g| \leq |n_s - 1|$, so $r \leq 8(1-n_s) \sim 0.3$ – confirmed by observations!

Duration of inflation

Duration of inflation was finite inside our past light cone. In terms of e-folds, difference in its total duration in different points of space can be seen by the naked eye from a smoothed CMB temperature anisotropy map.

ΔN formalism: $\Delta\zeta(\mathbf{r}) = \Delta N_{tot}(\mathbf{r})$ where $N_{tot} = \ln\left(\frac{a(t_{fin})}{a(t_{in})}\right) = N_{tot}(\mathbf{r})$ (AS, 1982,1985).

For $\ell \lesssim 50$, neglecting the Silk and Doppler effects, as well as the ISW effect due the presence of dark energy,

$$\frac{\Delta T(\theta,\phi)}{T_\gamma} = -\frac{1}{5}\Delta\zeta(r_{LSS},\theta,\phi) = -\frac{1}{5}\Delta N_{tot}(r_{LSS},\theta,\phi)$$

For $\frac{\Delta T}{T} \sim 10^{-5}$, $\Delta N \sim 5\times 10^{-5}$, and for $H \sim 10^{14}\,\mathrm{GeV}$, $\Delta t \sim 5 t_{Pl}$!

Inverse reconstruction of inflationary models in GR

In the slow-roll approximation:

$$\frac{V^3}{V'^2} = CP_\zeta(k(t(\phi))), \quad C = \frac{12\pi^2}{\kappa^6}$$

Changing variables for ϕ to $N(\phi)$ and integrating, we get:

$$\frac{1}{V(N)} = -\frac{\kappa^4}{12\pi^2}\int \frac{dN}{P_\zeta(N)}$$

$$\kappa\phi = \int dN\sqrt{\frac{d\ln V}{dN}}$$

Here, $N \gg 1$ stands both for $\ln(k_f/k)$ at the present time and the number of e-folds back in time from the end of inflation. First derived in H. M. Hodges and G. R. Blumenthal, Phys. Rev. D 42, 3329 (1990).

The two-parameter family of isospectral slow-roll inflationary models, but the second parameter shifts the field ϕ only.

Minimal "scale-free" reconstruction

Minimal inflationary model reconstruction avoiding introduction of any new physical scale both during and after inflation and producing the best fit to the Planck data.

Assumption: the numerical coincidence between $2/N_H \sim 0.04$ and $1 - n_s$ is not accidental but happens for all $1 \ll N \lesssim 60$: $P_\zeta = P_0 N^2$. Then:

$$V = V_0 \frac{N}{N + N_0} = V_0 \tanh^2 \frac{\kappa\phi}{2\sqrt{N_0}}$$

$$r = \frac{8 N_0}{N(N + N_0)}$$

$r \sim 0.003$ for $N_0 \sim 1$. From the upper limit on r:

$$N_0 < \frac{0.07 N^2}{8 - 0.07 N}$$

$N_0 < 57$ for $N = 57$.

Another example: $P_\zeta = P_0 N^{3/2}$.

$$V(\phi) = V_0 \frac{\phi^2 + 2\phi\phi_0}{(\phi + \phi_0)^2}$$

Not bounded from below (of course, in the region where the slow-roll approximation is not valid anymore). Crosses zero linearly.

More generally, the two "aesthetic" assumptions – "no-scale" scalar power spectrum and $V \propto \phi^{2n}$, $n = 1, 2...$ at the minimum of the potential – lead to $P_\zeta = P_0 N^{n+1}$, $n_s - 1 = -\frac{n+1}{N}$ unambiguously. From this, only $n = 1$ is permitted by observations. Still an additional parameter appears due to tensor power spectrum – no preferred one-parameter model (if the $V(\phi) \propto \phi^2$ model is excluded).

Inflation in $f(R)$ gravity

Purely geometrical realization of inflation.

The simplest model of modified gravity (geometrical primordial dark energy) considered as a phenomenological macroscopic theory in the fully non-linear and non-perturbative regime.

$$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\,d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R_\mu^\mu$$

Here $f''(R)$ is not identically zero. Usual matter described by the action S_m is minimally coupled to gravity.

Vacuum one-loop corrections depending on R only (not on its derivatives) are assumed to be included into $f(R)$. The normalization point: at laboratory values of R where the scalaron mass (see below) $m_s \approx \mathrm{const}$. Metric variation is assumed everywhere. Palatini variation leads to a different theory with a different number of degrees of freedom.

Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\,, \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

Reduction to the first order equation

In the absence of spatial curvature and $\rho_m = 0$, it is always possible to reduce these equations to a first order one using either the transformation to the Einstein frame and the Hamilton-Jacobi-like equation for a minimally coupled scalar field in a spatially flat FLRW metric, or by directly transforming the 0-0 equation to the equation for $R(H)$:

$$\frac{dR}{dH} = \frac{(R - 6H^2)f'(R) - f(R)}{H(R - 12H^2)f''(R)}$$

See, e.g. H. Motohashi amd A. A. Starobinsky, Eur. Phys. J. C **77**, 538 (2017), but in the special case of the $R + R^2$ gravity this was found and used already in the original AS (1980) paper.

Analogues of large-field (chaotic) inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R} \ , \ |A''(R)| \ll \frac{A(R)}{R^2} \ .$$

Analogues of small-field (new) inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1} \ , \ F''(R_1) \approx \frac{2F(R_1)}{R_1^2} \ .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

Perturbation spectra in slow-roll $f(R)$ inflationary models

Let $f(R) = R^2 A(R)$. In the slow-roll approximation $|\ddot{R}| \ll H|\dot{R}|$:

$$P_{\mathcal{R}}(k) = \frac{\kappa^2 A_k}{64\pi^2 A_k'^2 R_k^2}, \quad P_g(k) = \frac{\kappa^2}{12 A_k \pi^2}, \quad \kappa^2 = 8\pi G$$

$$N(k) = -\frac{3}{2}\int_{R_f}^{R_k} dR\, \frac{A}{A' R^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$.

Smooth reconstruction of inflation in $f(R)$ gravity

$$f(R) = R^2 A(R)$$

$$A = const - \frac{\kappa^2}{96\pi^2} \int \frac{dN}{P_{\mathcal{R}}(N)}$$

$$\ln R = const + \int dN \sqrt{-\frac{2\, d\ln A}{3\, dN}}$$

The additional assumptions that $P_{\mathcal{R}} \propto N^{\beta}$ and that the resulting $f(R)$ can be analytically continued to the region of small R without introducing a new scale, and it has the linear (Einstein) behaviour there, leads to $\beta = 2$ and the $R + R^2$ inflationary model with $r = \frac{12}{N^2} = 3(n_s - 1)^2$ unambiguously.

Conclusions

- At present, cosmology requires the introduction of four fundamental constants to describe observational data, additional to those known from laboratory physics.
- One new fundamental cosmological parameter $n_s - 1$ has been measured recently, but the theory had been able to predict it more than 30 years before the discovery.
- Regarding the present dark energy:
  a) still no statistically significant deviation from an exact cosmological constant;
  b) one constant is sufficient to describe its properties;
  c) no more than one new "coincidence problem".
- Regarding the primordial dark energy driving inflation in the early Universe:
  a number of inflationary models having only one free parameter can explain all existing observational data.

- The typical inflationary predictions that $|n_s - 1|$ is small and of the order of N_H^{-1}, and that r does not exceed $\sim 8(1 - n_s)$ are confirmed. Typical consequences following without assuming additional small parameters: $H_{55} \sim 10^{14}\,\mathrm{GeV}, \; m_{infl} \sim 10^{13}\,\mathrm{GeV}$.
- Though the Einstein gravity plus a minimally coupled inflaton remains sufficient for description of inflation with existing observational data, modified (in particular, scalar-tensor or $f(R)$) gravity can do it as well.
- From the scalar power spectrum $P_\zeta(k)$, it is possible to reconstruct an inflationary model both in the Einstein and $f(R)$ gravity up to one arbitrary physical constant of integration.

- In the Einstein gravity, the simplest inflationary models permitted by observational data are two-parametric, no preferred quantitative prediction for r, apart from its parametric dependence on $n_s - 1$, namely, $\sim (n_s - 1)^2$ or larger.
- In the $f(R)$ gravity, the simplest model is one-parametric and has the preferred value $r = \frac{12}{N^2} = 3(n_s - 1)^2$.
- Thus, it has sense to search for primordial GW from inflation at the level $r > 10^{-3}$!