Lectures on relativistic gravity and cosmology.

Lectures 13-14

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Possible candidates for DE apart from Λ

Small perturbations on FLRW background

Scalar perturbations

Vector and tensor perturbations

Gravitational instability in the matter dominated Universe

Possible candidates for DE apart from Λ

In the dispute between Plato and Democritus, Plato was right by 70%, and Democritus by 30%.

1. Physical DE: a scalar field with some interaction potential minimally coupled to gravity. Dubbed quintessence in the case of DE in the present Universe.

$$\mathcal{L}_\phi = \frac{1}{2}\phi_{,i}\phi^{,i} - V(\phi)\,.$$

If $\phi = \phi(t)$, then

$$\rho_{DE} = \frac{\dot\phi^2}{2} + V,\;\; p_{DE} = \frac{\dot\phi^2}{2} - V,\;\; w_{DE} + 1 = \frac{\dot\phi^2}{2V}\,.$$

Scalar field can mimic DE if $\dot\phi^2 \ll V(\phi)$ for a time period exceeding H^{-1}, and then $H^2 \approx \frac{8\pi G}{3}(\rho_m + V(\phi))$. For $V(\phi) = \frac{m_\phi^2\phi^2}{2}$, this requires $m_\phi \ll H$. DE phantom behaviour is not possible in this case.

2. Geometric DE: a specific form of $f(R)$ gravity $\mathcal{L}_g = \frac{f(R)}{2}$ in the range of R where $df/dR \approx 2f/R$.

3. Mixed DE: a scalar-tensor gravity.

$$\mathcal{L} = -\frac{1}{2}R\phi^2 + \frac{1}{2}\phi_{,i}\phi^{,i} - V(\phi)\,.$$

In the latter two cases, the effective gravitational constant depends on R or ϕ. Also DE phantom behaviour is possible.

Classification of small perturbations on a spatially flat FLRW background

The synchronous reference system is used.

$$ds^2 = dt^2 - a^2(t)(\delta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta, \ \alpha, \beta = 1, 2, 3.$$

The spatial dependence $e^{i\mathbf{kr}}$ is assumed, $k = |\mathbf{k}|$, index k for Fourier modes is omitted for brevity.

Types of perturbations:

1) scalar (otherwise dubbed adiabatic)

$$h_\alpha^\beta = \frac{1}{3}\mu(t)\,\delta_\alpha^\beta + \lambda(t)\left(\frac{1}{3}\delta_\alpha^\beta - \frac{k_\alpha k^\beta}{k^2}\right);$$

2) vector

$$h_\alpha^\beta = \nu(t)(s_\alpha k^\beta + s^\beta k_\alpha), \ s_\alpha k^\alpha = 0;$$

3) tensor (gravitational waves (GW))

$$h_\alpha^\beta = g(t)\,e_\alpha^\beta, \ e_\alpha^\beta k^\alpha = e_\alpha^\beta k_\beta = e_\alpha^\alpha = 0.$$

Scalar perturbations

Matter: an ideal barotropic fluid, $p = p(\rho)$, $c_s^2 = \frac{dp}{d\rho}$.

$$\lambda'' + 2\frac{a'}{a}\lambda' - \frac{k^2}{3}(\lambda + \mu) = 0, \ ' \equiv \frac{d}{d\eta} = a(t)\frac{d}{dt} ,$$

$$\mu'' + \frac{a'}{a}(2 + 3c_s^2)\,\mu' + \frac{k^2}{3}(1 + 3c_s^2)(\lambda + \mu) = 0 ,$$

$$\delta\rho_{syn} = \frac{1}{24\pi G a^2}\left[k^2(\lambda + \mu) + \frac{3a'}{a}\mu'\right] .$$

Matter flow is potential: $u_\alpha = -\frac{ik_\alpha V_{syn}}{a}$, $V_{syn} = -\frac{\lambda' + \mu'}{24\pi G a^2(\rho + p)}$.
The synchronous reference system does not fix all freedom of coordinate transformations (gauge freedom). Thus, these equations always have two gauge solutions:

$$1)\,\lambda = -\mu = const, \ 2)\,\lambda = -k^2\int\frac{d\eta}{a}, \ \mu = k^2\int\frac{d\eta}{a} - \frac{3a'}{a^2} .$$

Longitudinal gauge

$$ds^2 = (1 + 2\Phi)\, dt^2 - (1 - 2\Psi)\, a^2(t)(dx^2 + dy^2 + dz^2).$$

Valid in the linear order only. Gauge freedom is fixed completely.

$$\Phi = -\frac{1}{2k^2}(\lambda'' + \frac{a'}{a}\lambda'),\ \ \Psi = -\frac{1}{6}(\lambda + \mu) + \frac{a'}{2k^2 a}\,\lambda'\,.$$

For an ideal barotropic fluid, and more generally, if $\delta T_\alpha^\beta = 0$ for $\alpha \neq \beta$,

$$\Phi = \Psi\,.$$

In this case we get the following equation (the generalized Poisson equation) for the quantity $\delta\rho = \delta\rho_{syn} + 3(\rho + p)\frac{a'}{a} V_{syn}$ that represents the comoving density perturbation (the density perturbation in the reference system where $u^\alpha = 0$):

$$-\frac{k^2}{a^2}\,\Phi = 4\pi G \delta\rho\,.$$

Master equation for Φ

The master equation for one-component fluid (including Λ, if non-zero):

$$\ddot{\Phi} + (4 + 3c_s^2)\, H\dot{\Phi} + \left[\frac{c_s^2 k^2}{a^2} + 3H^2\left(c_s^2 - \frac{p}{\rho}\right)\right]\Phi = 0\,.$$

Another gauge-invariant quantity (dubbed curvature perturbation) for which the master equation can be obtained:

$$\mathcal{R} = -\zeta = \frac{1}{6}\left[\lambda + \mu - \frac{H}{\dot{H}}\,(\dot{\lambda} + \dot{\mu})\right]\,.$$

The characteristic feature of a gauge-invariant quantity: it is nullified by gauge solutions.

The characteristic scale $L_J = c_s H^{-1}$, sometimes called the Jeans length. Two characteristic regimes:
1) the long-wave regime $a(t)/k \gg L_J,\ c_s k \ll aH,\ c_s k\eta \ll 1$;
2) the short-wave regime $a(t)/k \ll L_J,\ c_s k \gg aH,\ c_s k\eta \gg 1$.

The long-wave regime

Early time behaviour for all scales in the case of expansion law $a(t) \propto t^q, \ q < 1$ that happens for $p > -\rho/3$. Then the solution for $k = 0$ can be used in the first approximation.

Modern way of finding solutions for long-wave inhomogeneous perturbations: variation of the background solution with respect to its parameters. Solutions without equations.

For $\mathcal{K} = 0$, $a = a_0 f(t)$. a_0 does not appear in $\rho(t)$. Thus,

$$\mu = 6\mathcal{R} = 6\,\frac{\delta \ln a(t)}{\delta a_0} = const, \ \lambda = 0, \ \delta\rho_{syn} = 0$$

is one of the solutions in the limit $c_s k \ll aH$ (a constant λ can be put zero using one of the gauge solutions).

Next order in k^2 correction:

$$\mu = 6\mathcal{R} + k^2\mu_1(t),\ \lambda = k^2\lambda_1(t)\,,$$

$$\lambda_1' = \frac{2\mathcal{R}}{a^2}\int a^2\,d\eta,\ \mu_1 = -2\mathcal{R}\,\frac{a'}{a^2}\int\frac{a^3\,d\eta}{a'^2}\,,$$

$$\Phi = \Psi = \mathcal{R}\left(1 - \frac{a'}{a^3}\int a^2\,d\eta\right) = \mathcal{R}\left(1 - \frac{H}{a}\int a\,dt\right).$$

For $a \to 0$ and $\eta \to 0$, one of solutions for metric perturbations is finite (the quasi-isotropic, or (density) growing mode), while the other diverges (the decaying mode).

It is straightforward to check that these solutions satisfy the master equation for Φ with $k = 0$ using the identities for the background evolution:

$$\dot{H} = -4\pi G(\rho + p),\ 1 + \frac{p}{\rho} = -\frac{2\dot{H}}{3H^2},\ c_s^2 = -1 - \frac{\ddot{H}}{3H\dot{H}}\,.$$

Vector perturbations

Matter is assumed to be an ideal barotropic fluid.

$$\nu'' + 2\frac{a'}{a}\nu = 0,\ \nu' \propto a^{-2},\ \nu \propto \int d\eta\, a^{-2} = \int dt\, a^{-3},\ \delta\rho = 0\,.$$

Since $\nu = const$ is a gauge solution, vector perturbations decay with time and diverge at $t \to 0$ if $p = \alpha\rho$ with $\alpha \leq 1,\ c_s \leq 1.$ Time behaviour of ν corresponds to the conservation of the angular momentum.

Tensor perturbations

$$g'' + 2\frac{a'}{a}g' + k^2 g = 0,\ \delta\rho = \delta u^\alpha = 0\,.$$

The same equation as for a massless scalar field. The long-wave (super-Hubble, $k\eta \ll 1$) behaviour:

$$g = g_1 + g_2 \int d\eta\, a^{-2}\,.$$

Constant (quasi-isotropic) and decaying modes. The existence of the constant mode follows from the possibility of arbitrary rescaling of spatial coordinates x, y, z in an arbitrary background FLRW solution keeping the total spatial volume fixed.

The short-wave (sub-Hubble) behaviour: $g \propto a^{-1}\exp(ik\eta)$. GW moving with the light velocity and with the amplitude decreasing due to the Universe expansion (redshift). No instability.

Quasi-isotropic initial conditions in the early Universe

From 8 arbitrary functions of **k**, or **r** in the coordinate representation, in the initial conditions for perturbations at the singularity $a \to 0, \; t \to 0$, three (one for scalar and two for tensor perturbations) remains finite and do not destroy isotropy and homogeneity of the early Universe, while the other five (one for scalar, two for vector and two for tensor perturbations) diverge and destroy isotropy and homogeneity.

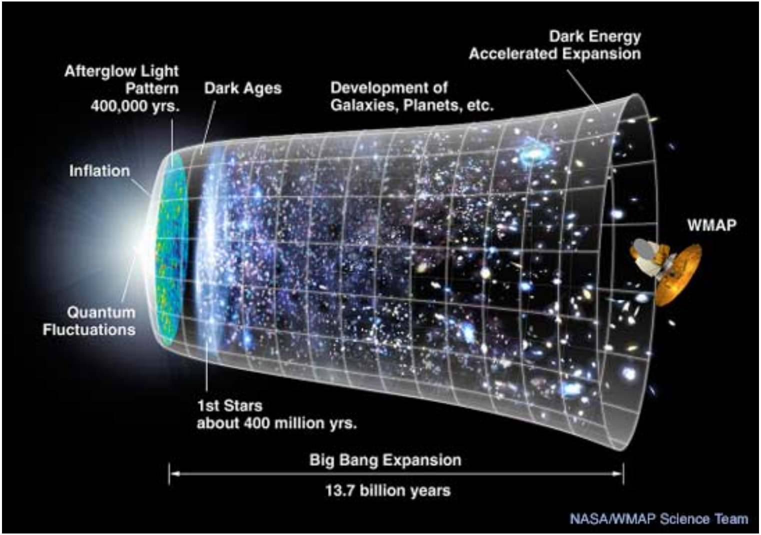
Thus, if we are sure that the Universe was isotropic and homogeneous at sufficiently early time in the past, the latter five functions describing decaying modes of perturbations should be zero (or very small).

Why we are sure:

1) for $L \gtrsim 10$ Mpc - from the observed CMB temperature isotropy at recombination ($z \approx 1100$);
2) for $L \gtrsim 100$ pc - from the BBN ($z = (10^9 - 10^{10})$).

The theory - viable models of the inflationary scenario of the early Universe - explains the absence of decaying modes of perturbations for $L \gtrsim 1$ cm - practically, for all scales. According to this theory, the Universe was (approximately) isotropic and homogeneous from the very early time in the past.

From the remaining 3 functions $\Phi(\mathbf{r}),\ g_{1,2}(\mathbf{r})$, only the first one is responsible for the observed distribution of visible matter in the Universe. Properties of all these 3 functions are predicted by any concrete viable inflationary model of the early Universe.


Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years
NASA/WMAP Science Team

Linear gravitational instability in the matter dominated Universe

The matter dominated stage began at $1 + z = \frac{\Omega_m}{\Omega_{rad}} \approx 3540$ for $H_0 = 70$ km/s/Mpc and $\Omega_m = 0.3$, where $\Omega_{rad} = \Omega_\gamma + \Omega_\nu \approx 1.68\,\Omega_\gamma$.

$$1.68 = 1 + 3 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{4/3} = 1 + 3 \cdot 0.227\,.$$

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The growing mode in the case of dust ($c_s = 0$)

$$a(t) \propto t^{2/3},\ \rho = \frac{1}{6\pi G t^2}\,,$$

$$\Phi = -\frac{3}{5}\,\mathcal{R} = const,\ \ V = \Phi t,\ \delta \equiv \frac{\delta\rho}{\rho} = -\frac{3k^2 t^2 \Phi}{2a^2} \propto t^{2/3}\,,$$

where V is the velocity potential in the longitudinal gauge.

The typical value of $\mathcal{R}$ and Φ is $\sim 10^{-5}$ at all cosmological scales as follows from CMB temperature anisotropy and polarization. In the absence of CDM, growth of δ beyond this value would began after recombination at $z = z_{rec} \approx 1100$ only, when baryons and photons decouple. Thus, this would not be sufficient for formation of galaxies, stars, planets, etc. by the present time.

Indeed, if c_s is not very small, then for $L \ll L_J, \ kc_s \gg aH$,

$$\Phi \propto \sqrt{\frac{\rho + p}{c_s}} \exp\left(\pm ik \int c_s \, d\eta\right), \ \delta \propto \frac{k^2 \Phi}{G a^2 \rho} \sim \frac{t}{a^2} \, .$$

Sound velocity for adiabatic perturbations (acoustic waves) in the photon-baryon plasma:

$\rho = \rho_b + \rho_\gamma, \ p = \frac{1}{3}\rho_\gamma, \ \rho_b = m_p n_b, \ \frac{dn_b}{n_b} = -\frac{dV}{V}, \ \frac{d\rho_\gamma}{\rho_\gamma} = -\frac{4}{3}\frac{dV}{V},$

$$c_s^2 = \frac{dp}{d\rho} = \frac{d\rho_\gamma}{3(d\rho_b + d\rho_\gamma)} = \frac{1}{3\left(1 + \frac{3}{4}\frac{\rho_b}{\rho_\gamma}\right)} \, .$$

Generalization to the presence of unclustered component

Cold matter (baryons + CDM) with $\rho = \rho_m(t) \propto a^{-3}$ and unclustered component (Λ, dark energy, hot dark matter, radiation) for which $L \ll L_J$ (or free streaming length in case of free particles).

Linear quasi-Newtonian hydrodynamics:

$$\frac{\triangle\Phi}{a^2} = 4\pi G\rho_m\delta, \;\; \dot{\delta} + \frac{\mathrm{div}\,\mathbf{u}}{a} = 0, \;\; \mathbf{u} = -\frac{\nabla V}{a}, \;\; (a\mathbf{u})^{\cdot} = -\nabla\Phi\,.$$

Potential flow: measuring of v_r determines V and $\mathbf{u}$.

Observations confirm the absence of rotational peculiar velocities of galaxies for $L \gtrsim 10$ Mpc.

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_m\delta = 0\,.$$

Home task.

Let the unclustered component has $p = 0$, so $\rho_m = \frac{\Omega_m}{6\pi G t^2}$, $\Omega_m = const$ during the matter dominated stage. Find $\delta(t)$.

Another application: slow growth of density perturbations in CDM at all scales during the radiation dominated stage when $\rho_m \ll \rho_{rad}$, $a(t) \propto t^{1/2}$.

$$\delta \propto \ln t\,.$$

Inversion of the equation for δ gives the reconstruction of $H(z)$ from $\delta(z)$:

$$\frac{H^2}{H_0^2} = \frac{3\Omega_m a_0^3}{a^6}\left(\frac{d\delta}{da}\right)^{-2}\int_0^a a\delta\,\frac{d\delta}{da}\,da =$$

$$= 3\Omega_m(1+z)^2\left(\frac{d\delta}{dz}\right)^{-2}\int_z^\infty \delta\,\left|\frac{d\delta}{dz}\right|\,\frac{dz}{1+z} =$$

$$= \frac{(1+z)^2\left(\frac{d\delta}{dz}\right)^2_{z=0}}{\left(\frac{d\delta}{dz}\right)^2} - 3\Omega_m(1+z)^2\left(\frac{d\delta}{dz}\right)^{-2}\int_0^z \delta\,\left|\frac{d\delta}{dz}\right|\,\frac{dz}{1+z}\,.$$