Lectures on relativistic gravity and cosmology.

Lectures 3-4

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Special relativity electrodynamics

Non-inertial reference systems

The equivalence principle

The Riemann tensor

The vacuum Einstein equations with a cosmological constant

The Schwarzschild metric

Electromagnetic interactions

New physical quantity: electric charge e. Quantized, $\pm n$ for free particles (protons, electrons, etc.), $\pm \frac{n}{3}$ for quarks.

The fine structure constant $\alpha = \frac{e^2}{\hbar c}$. The 2018 CODATA recommended value
$\alpha^{-1} = 137.035999084(21)$.

No change in the Maxwell equations: they are already invariant under the Lorentz transformation.
The 4-vector potential: $A^i = (\phi, \mathbf{A})$.
The electromagnetic field tensor: $F_{ik} = A_{k,i} - A_{i,k}$.
In the 3D form:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\ \phi, \ \mathbf{H} = \mathrm{rot}\ \mathbf{A}$$

$$E_x = F_{01},\ E_y = F_{02},\ E_z = F_{03},\ H_x = F_{32},\ H_y = F_{13},\ H_z = F_{21}$$

Gauge transformations: $A_k \to A'_k = A_k - \frac{\partial f}{\partial x^k}$.

Action and equations for a charge

Action for a point massive charge:

$$S = -mc\int_a^b ds - \frac{e}{c}\int_a^b A_i dx^i$$

Equations of motion:

$$mc\frac{du^i}{ds} = \frac{e}{c}F^{ik}u_k$$

In the 3D form:

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{vH}], \quad \frac{d\mathcal{E}}{dt} = e\mathbf{Ev}$$

where $\mathcal{E}$ and **p** are the same as in the absence of electromagnetic field.

Action and equations for the field

Action or the electromagnetic field:

$$S_{em} = -\frac{1}{16\pi c}\int F_{ik}F^{ik}d\Omega, \quad d\Omega = c\,dt\,dx\,dy\,dz$$

The Maxwell equations (comma - partial derivative):

$$F_{ik,l} + F_{li,k} + F_{kl,i} = 0$$

$$F^{ik}_{\ ,k} = \frac{4\pi}{c}j^i$$

The electric 4-current $j^i = (c\rho, \mathbf{j})$.

Testing the constancy of α.

1. Laboratory measurements (2008):
$\frac{\dot{\alpha}}{\alpha} = (-1.6 \pm 2.3) \times 10^{-17}$ per year.
2. Measurements using remote quasars: $|\frac{\Delta\alpha}{\alpha}| \lesssim 10^{-5}$ for the last 10-12 billion years.

Non-inertial reference frames (NIRF)

General transformation of coordinates, 4-vectors and 4-tensors:

$$x^i = x^i(\tilde{x}^k),\ dx^i = \frac{\partial x^i}{\partial \tilde{x}^k}\, d\tilde{x}^k,\ A^i = \frac{\partial x^i}{\partial \tilde{x}^k}\, \tilde{A}^k,\ A_i = \frac{\partial \tilde{x}^k}{\partial x^i}\, \tilde{A}_k$$

$$A^{ik} = \frac{\partial x^i}{\partial \tilde{x}^l}\frac{\partial x^k}{\partial \tilde{x}^m}\, \tilde{A}_{lm},\ A_{ik} = \frac{\partial \tilde{x}^l}{\partial x^i}\frac{\partial \tilde{x}^m}{\partial x^k}\, \tilde{A}_{lm},\ A^i{}_k = \frac{\partial x^i}{\partial \tilde{x}^l}\frac{\partial \tilde{x}^m}{\partial x^k}\, \tilde{A}^l{}_m$$

δ^i_k - 4-tensor. The metric tensor: $ds^2 = g_{ik}dx^i dx^k$. Properties: $g_{ik} = g_{ki},\ g_{il}g^{kl} = \delta^i_k,\ g \equiv \mathrm{Det}(g_{ik}) < 0,\ A^i = g^{ik}A_k,\ A_i = g_{ik}A^k$.

Transformation of the 4-volume differential:

$$J = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} = \frac{1}{\sqrt{-g}},\ \ d\tilde{\Omega} \to \frac{1}{J}\, d\Omega = \sqrt{-g}\, d\Omega$$

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An important particular case: transformation to an uniformly rotating reference system:

$$x = \tilde{x}\cos\Omega t - \tilde{y}\sin\Omega t,\ \ y = \tilde{x}\sin\Omega t + \tilde{y}\cos\Omega t,\ \ z = \tilde{z}$$

$$ds^2 = \left[c^2 - \Omega^2(\tilde{x}^2 + \tilde{y}^2)\right] dt^2 - d\tilde{x}^2 - d\tilde{y}^2 - d\tilde{z}^2$$

$$+2\Omega\tilde{y}\, d\tilde{x}\, dt - 2\Omega\tilde{x}\, d\tilde{y}\, dt$$

Light cylinder: $\tilde{x}^2 + \tilde{y}^2 = \frac{c^2}{\Omega^2}$. The reference system may not be realized by rigid physical bodies beyond it.

Covariant partial derivative

Semicolon - covariant partial derivative.
$\frac{\partial a}{\partial x^i} \equiv a_{,i} = a_{;i}$, but $A^i_{,k}$ is not a 4-tensor and $dA^i = A^i(x^k + dx^k) - A^i(x^k)$ is not a 4-vector. To determine the difference of two 4-vectors, they should be first placed into one point of space-time by using the vector parallel transport operation. The change of a 4-vector A^i after its parallel transport from x^l to $x^l + dx^l$ is given by:

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l$$

where Γ^i_{kl} are the Christoffel symbols (or, the affine connection). By considering the transport of the scalar product, $\delta(A_i B^i) = 0$, we get $\delta A_i = \Gamma^k_{il} A_k dx^l$.

The total covariant change in A^i and A_i:

$$DA^i = dA^i - \delta A^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k\right) dx^l$$

$$DA_i = dA_i - \delta A_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k\right) dx^l$$

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k, \; A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k$$

The assumptions:
1) $g_{ik;l} = 0$ - no nonmetricity;
2) $\Gamma^i_{kl} = \Gamma^i_{lk}$ - no torsion.
Then the Christoffel symbols coincide with the Levi-Civita connection. Let us define $\Gamma_{i,kl} = g_{im} \Gamma^m_{kl}$ (here comma does not mean derivative).

Relation of the Levi-Civita connection to metric

From $g_{ik;l} = 0$:

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,il} + \Gamma_{i,kl}, \ \frac{\partial g_{il}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik}, \ -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,il}$$

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right)$$

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right)$$

Useful relations:

$$dg = g g^{ik} dg_{ik} = -g g_{ik} dg^{ik}, \ \Gamma^k_{ik} = \frac{\partial \ln \sqrt{-g}}{\partial x^i}$$

$$A^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} A^i)}{\partial x^i}, \ \phi^{;i}_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i}\left(\sqrt{-g} g^{ik} \frac{\partial \phi}{\partial x^k}\right)$$

The equivalence principle

The gravitational constant (the 2018 CODATA recommended value) $G = 6.67430(15) \times 10^{-8}\ \mathrm{cm}^3\ \mathrm{g}^{-1}\ \mathrm{s}^{-2}$.
Relativistic effects in gravity becomes important when $\frac{|\Delta\varphi|}{c^2}$ is not small compared to unity where φ is the Newtonian gravitational potential. Further in my lectures: $c = 1$.

Two forms of the equivalence principle.
1. The weak equivalence principle (WEP): $m_i = m_{g,pass}$.
Tested with the accuracy 10^{-14} by now (the MICROSCOPE mission, arXiv:1712.01176).
2. The strong equivalence principle (SEP): $m_{g,pass} = m_{g,act}$.
Tested with the accuracy 3×10^{-6} by now (in the triple stellar system PSR J0337+1715 : a neutron star (radio pulsar) and two white dwarfs, arXiv:1807.02059).

Action and equations of motion for a massive particle

WEP: locally motion in the gravitational field is indistinguishable from that in a NIRF. But now we don't assume that the interval *ds* can be globally transformed to its form in the Minkowski space-time. Locally, it is always possible. Moreover, locally all the Christoffel symbols can be made zero using the local coordinate transformation $\tilde{x}^i = x^i + \frac{1}{2}(\Gamma^i_{kl})_0 x^k x^l$ - the local IRF.

The action has the same form as in Special Relativity:

$$S = -mc \int_a^b ds$$

Equations of motion:

$$\frac{Du^i}{ds} \equiv \frac{du^i}{ds} + \Gamma^i_{kl} u^k u^s = 0$$

The Hamilton-Jacobi equation has the same form, too:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = m^2c^2$$

The limit $m = 0$ gives the eikonal equation in GR for the motion of light in the geometric optics approximation:

$$g^{ik}\frac{\partial \psi}{\partial x^i}\frac{\partial \psi}{\partial x^k} = 0$$

Transition to the non-relativitic Newton gravity:

$$L = -mc^2 + \frac{mv^2}{2} - m\varphi \to ds = \left(1 - \frac{v^2}{2} + \varphi\right) dt \to$$

$$\to ds^2 = (1 + 2\varphi)dt^2 - d\mathbf{r}^2 \to g_{00} = 1 + 2\varphi$$

The Riemann tensor

The new effect appearing in curved space-time: a 4-vector changes after its parallel transport along a closed loop $\Delta A_k = \frac{1}{2} R^i{}_{klm} A_i \Delta f^{lm}$.
An equivalent way to introduce the Riemann tensor (the second conventional choice of sign):

$$A_{i;k;l} - A_{i;l;k} = A_m R^m{}_{ikl}$$

$$R^i{}_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}$$

Properties of the Riemann tensor:

1. $R_{iklm} = -R_{kilm} = -R_{ikml} = R_{lmik}$.
2. $R_{iklm} + R_{imkl} + R_{ilmk} = 0$.
3. $R^n{}_{ikl;m} + R^n{}_{imk;l} + R^n{}_{ilm;k} = 0$ - the Bianchi identity.

The Ricci tensor and the gravitational field action

The Ricci tensor is defined as (the third conventional choice of sign):

$$R^i_k = R^m{}_{imk}$$

$$R_{ik} = R_{ki}, \;\; R^k_{i;k} = \frac{1}{2} R_{,i}$$

The action for the gravitational field:

$$S_g = -\frac{1}{16\pi G} \int (R + 2\Lambda)\sqrt{-g}\, d\Omega$$

The action can be represented in a not generally covariant form containing squares of the first derivatives of the metric tensor using the identity

$$\sqrt{-g}R = \sqrt{-g}g^{ik}(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}) + \frac{\partial(\sqrt{-g}w^i)}{\partial x^i}$$

The vacuum Einstein equations with a cosmological constant

$$\delta S_g/\delta g^{ik} = 0$$

The useful relation $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\,\delta g = -\frac{1}{2}\sqrt{-g}\,g_{ik}\delta g^{ik}$.

$(-16\pi G)\delta S_g =$
$\int\left(R_{ik} - \frac{1}{2}g_{ik}(R + 2\Lambda)\right)\delta g^{ik}\sqrt{-g}\,d\Omega + \int g^{ik}\delta R_{ik}\sqrt{-g}\,d\Omega$

It can be shown that the last term in the integrand is the total derivative:

$$\sqrt{-g}\,g^{ik}\delta R_{ik} = \frac{\partial}{\partial x^l}(\sqrt{-g}\,v^l)$$

In the local IRF where all $\Gamma^i_{kl} = 0$, $v^l = g^{ik}\delta\Gamma^l_{ik} - g^{il}\delta\Gamma^k_{ik}$.
The vacuum Einstein equations with a cosmological constant

$$R_{ik} - \frac{1}{2}g_{ik}R = g_{ik}\Lambda$$

The differentiation $(;k)$ gives $0 = 0$ due to the Bianchi identity.

Spherically symmetric vacuum gravitational field

The most general spherically symmetric space-time metric:

$$ds^2 = A(t,r)dt^2 + B(t,r)dtdr - C(t,r)dr^2 - D(t,r)(d\theta^2 + \sin^2\theta d\phi^2)$$

Remaining freedom of coordinate transformations:
$r = f_1(\tilde{r}, \tilde{t}),\ t = f_2(\tilde{r}, \tilde{t})$.

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\Gamma^1_{11} = \frac{\lambda'}{2},\ \Gamma^0_{10} = \frac{\nu'}{2},\ \Gamma^1_{00} = \frac{\nu'}{2}e^{\nu-\lambda}$$

$$\Gamma^0_{00} = \frac{\dot{\nu}}{2},\ \Gamma^1_{10} = \frac{\dot{\lambda}}{2},\ \Gamma^0_{11} = \frac{\dot{\lambda}}{2}e^{\lambda-\nu}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^1_{22} = -re^{-\lambda},\ \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta$$

$$\Gamma^3_{23} = \cot\theta,\ \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Lambda = 0, \; R_1^1 - \frac{1}{2}R = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 0$$

$$R_0^0 - \frac{1}{2}R = -e^{-\lambda}\left(-\frac{\lambda'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 0, \; R_0^1 = -e^{-\lambda}\frac{\dot{\lambda}}{r} = 0$$

From the last equation: $\lambda = \lambda(r)$. The difference of the second and third equations gives $\lambda + \nu = f(t)$. $f(t)$ can be made zero by using the remaining possible transformation of time $t = f_3(\tilde{t})$. Thus, spherically symmetric vacuum gravitational field appears to be static (for the range of r for which it has sense). Then integration of the second equation gives $e^{-\lambda} = e^{\nu} = 1 - \frac{r_g}{r}, \; r_g = 2GM$.

The Schwarzschild metric describing the outer region of a non-rotating black hole:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Space-time remains regular at $r = r_g$.