Lectures on relativistic gravity and cosmology.

Lectures 5-6

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Light motion in the Schwarzschild metric

The Kerr metric

The matter metric energy-momentum tensor

The Einstein equations with matter

Some physical numbers and black holes

$R_\odot = 6.963 \cdot 10^{10}\,\mathrm{cm},\;\; M_\odot = 1.989 \cdot 10^{33}\mathrm{g},\;\; r_{g,\odot} \approx 3\,\mathrm{km}$

$1\,\mathrm{au} = 1.495978707 \cdot 10^{13}\,\mathrm{cm}$

$1\,\mathrm{pc} = 3.086 \cdot 10^{18}\,\mathrm{cm} = 206265\,\mathrm{au} \approx 3.26\,\mathrm{ly}$

Two types of black holes.

1) Stellar mass black holes $M = (3 - 150)\ M_\odot$
a) in X-ray double stellar systems,
b) in BH merger events seen by GW antennas.

2) Supermassive black holes is galaxy centres and quasars
$M = (10^6 - 6 \cdot 10^{10})\ M_\odot$.

Black hole in the center of our Galaxy (Milky Way)

Sagittarius A* (Sgr A*).
Distance from us $\approx 8\,\mathrm{kpc}$.
$M \approx 4.3 \cdot 10^6\, M_\odot,\ r_g \approx 0.09\,\mathrm{au}$.
Determined by observation of star motion around it for more than 20 years (R. Genzel and A. Ghez). Stars with the closest approach:
1) $S4714$, $T = 12\,\mathrm{y},\ R_{pc} \approx 13\,\mathrm{au},\ e = 0.985$;
2) $S2$, $T = 16\,\mathrm{y},\ R_{pc} \approx 120\,\mathrm{au},\ e = 0.884,\ \delta\phi = 16'$ per orbital period.

Light motion in the Schwarzschild metric

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k}=0$$

$$ds^2=\left(1-\frac{r_g}{r}\right)dt^2-\frac{dr^2}{1-\frac{r_g}{r}}-r^2(d\theta^2+\sin^2\theta d\phi^2)$$

For motion in the equatorial plane:

$$\psi=-\omega t+L\phi+\psi_r(r),\ \psi_r=\int\frac{dr}{1-\frac{r_g}{r}}\sqrt{\omega^2-L^2U(r)}$$

$$U(r)=\left(1-\frac{r_g}{r}\right)\frac{1}{r^2}$$

$L=b\omega$ where b is the impact parameter. The light trajectory

$$\frac{\partial\psi}{\partial L}=\mathrm{const},\ \phi=\int\frac{dr}{r^2\sqrt{b^{-2}-U(r)}}$$

For $r_g=0$, $r=\frac{b}{\cos\phi}$.

Light deflection at large distances

Let $r \gg r_g, b \gg r_g$.
Let us change the radial variable $r \to r + \frac{r_g}{2}$ and expand the radial part of the eikonal in powers of r_g/r.

$$\psi_r = \omega \int dr \sqrt{1 + \frac{2r_g}{r} - \frac{b^2}{r^2}} = \psi_r^0 + \omega r_g \int \frac{dr}{\sqrt{r^2 - b^2}}$$

The total change of the eikonal when the light beam moves from $r = R$ to $r = b$ and back to $r = R$ is

$$\Delta\psi_r = \Delta\psi_r^0 + 2\omega r_g \int_b^R \frac{dr}{\sqrt{r^2 - b^2}}$$

The light deflection angle is

$$\Delta\phi = -\frac{\partial \Delta\psi_r}{\partial L} = -\frac{\partial \Delta\psi_r^0}{\partial L} + \frac{2r_g R}{b\sqrt{R^2 - b^2}}$$

In the limit $R \to \infty$, $\Delta\phi = \pi + \delta\phi$, where $\delta\phi = \frac{2r_g}{b}$.
For Sun ($b = R_\odot$), $\delta\phi = 1.75''$. Tested with the 10^{-5} accuracy.

Photon sphere and black hole shadow

The unstable photon orbit: $U'(r) = 0 \to r = r_1 = \frac{3}{2} r_g$. The corresponding impact parameter

$$b = U(r_1)^{-1/2} = \frac{3\sqrt{3}}{2}\, r_g$$

.

This impact parameter gives the answer to the two problems:
1) the absorption cross-section of ultrarelativistic particles by a black hole $\sigma_{abs} = \pi b^2$.
2) the radius of the black hole shadow.

arXiv:1906.11238: observation of the shadow of the black hole in the M87 galaxy with

$$M = (6.5 \pm 0.7) \cdot 10^9\, M_\odot, \;\; r_g \approx 2 \cdot 10^{15}\,\mathrm{cm} \approx 130\,\mathrm{au}$$

and distance from the Sun $R \approx 17\,\mathrm{Mpc}$.

The Kerr metric

The stationary solution of the Einstein equations in vacuum.

$$ds^2 = \left(1 - \frac{r_g r}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 -$$

$$- \left(r^2 + a^2 + \frac{r_g a^2 r}{\rho^2} \sin^2\theta\right) \sin^2\theta \, d\phi^2 + \frac{2 r_g a r}{\rho^2} \sin^2\theta \, dt d\phi \ ,$$

$$\Delta = r^2 - r_g r + a^2, \ \rho^2 = r^2 + a^2 \cos^2\theta, \ \sqrt{-g} = \rho^2 \sin\theta \ ,$$

$$r_g = 2GM, \ L = Ma, \ 0 \le a \le GM$$

The space-time becomes flat for $M = 0$.

The event horizon: $g^{11} = 0, \ r_+ = GM + \sqrt{G^2 M^2 - a^2}$.
The outer surface of the ergosphere:
$g_{00} = 0, \ r_e = GM + \sqrt{G^2 M^2 - a^2 \cos^2\theta}$.
Geodesic particles inside the ergosphere should corotate with the black hole.

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = \frac{1}{\Delta}\left(r^2 + a^2 + \frac{r_g a^2 r}{\rho^2}\sin^2\theta\right)\left(\frac{\partial S}{\partial t}\right)^2 - \frac{\Delta}{\rho^2}\left(\frac{\partial S}{\partial r}\right)^2 -$$

$$-\frac{1}{\rho^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{1}{\Delta\sin^2\theta}\left(1 - \frac{r_g r}{\rho^2}\right)\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{2r_g a r}{\rho^2\Delta}\frac{\partial S}{\partial t}\frac{\partial S}{\partial \phi}$$

. Separation of variables in the Hamilton-Jacobi and eikonal equations for an arbitrary motion.

The Christodoulou formula:

$$M^2 = M_{ir}^2 + \frac{L^2}{4M_{ir}^2}$$

where $M_{ir}^2 = \frac{Mr_+}{2G}$, $S_{hor} = 16\pi M_{ir}^2$. M_{ir} cannot decrease with time in classical General Relativity (the Second Law of the Black Hole Thermodynamics).

Killing equations

Let us find infinitesimal transformations of coordinates $\tilde{x}^i = x^i + \xi^i$ which do not change the space-time metric.

$$\tilde{g}^{ik}(\tilde{x}^l) = g^{lm}(x^l)\frac{\partial \tilde{x}^i}{\partial x^l}\frac{\partial \tilde{x}^k}{\partial x^m} = g^{lm}\left(\delta^i_l + \frac{\partial \xi^i}{\partial x^l}\right)\left(\delta^k_m + \frac{\partial \xi^k}{\partial x^m}\right) \approx$$

$$\approx g^{ik}(x^l) + g^{im}\frac{\partial \xi^k}{\partial x^m} + g^{kl}\frac{\partial \xi^i}{\partial x^l} \ .$$

. On the other hand,

$$\tilde{g}^{ik}(\tilde{x}^l) = \tilde{g}^{ik}(x^l + \xi^l) = \tilde{g}^{ik}(x^l) + \xi^l \frac{\partial \tilde{g}^{ik}}{\partial x^l} \ ,$$

$$\tilde{g}^{ik}(x^l) = g^{ik}(x^l) - \xi^l \frac{\partial \tilde{g}^{ik}}{\partial x^l} + g^{il}\frac{\partial \xi^k}{\partial x^l} + g^{kl}\frac{\partial \xi^i}{\partial x^l} = g^{ik}(x^l) + \xi^{i;k} + \xi^{k;i} \ ,$$

$$\tilde{g}_{ik} = g^{ik} - \xi_{i;k} - \xi_{k;i} \ .$$

The Killing equations

$$\xi^{i;k} + \xi^{k;i} = 0$$

The matter metric energy-momentum tensor

The matter action $S = \int \mathcal{L}_m \sqrt{-g}\, d\Omega$.

$$\delta S = \int \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{ik}} \delta g^{ik} + \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial(\partial g^{ik}/\partial x^l)} \delta \frac{\partial g^{ik}}{\partial x^i} \right) d\Omega =$$

$$= \int \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l} \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial(\partial g^{ik}/\partial x^l)} \right) \delta g^{ik} d\Omega \; .$$

Definition of the matter metric energy-momentum tensor (EMT)

$$\frac{1}{2}\sqrt{-g}\, T_{ik} = \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l} \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial(\partial g^{ik}/\partial x^l)} \; .$$

Then $\delta S = \frac{1}{2} \int T_{ik} \delta g^{ik} \sqrt{-g}\, d\Omega = -\frac{1}{2} \int T^{ik} \delta g_{ik} \sqrt{-g}\, d\Omega$.

$$\delta S = \frac{1}{2} \int T_{ik} (\xi^{i;k} + \xi^{k;i}) \sqrt{-g}\, d\Omega = \int T_{ik} \xi^{i;k} \sqrt{-g}\, d\Omega$$

Note that δg_{ik} are not independent here!

Conservation of the matter metric EMT

$\delta S = \int \left((T_i^k \xi^i)_{;k} - T^k_{i;k} \xi^i \right) \sqrt{-g}\, d\Omega = -\int T^k_{i;k} \xi^i \sqrt{-g}\, d\Omega$,
since $\sqrt{-g} A^i_{;i} = \frac{\partial(\sqrt{-g} A^i)}{\partial x^i}$.

$$\delta S = 0 \rightarrow T^k_{i:k} = 0$$

1. EMT for the electromagnetic field:

$$\mathcal{L}_m = -\frac{1}{16\pi} F_{ik} F_{lm} g^{il} g^{km}$$

$$T_{ik} = \frac{1}{4\pi} \left(-F_{il} F_k{}^l + \frac{1}{4} F_{lm} F^{lm} g_{ik} \right), \quad T = 0 \ .$$

2. EMT for a barotropic fluid (or gas):

$$T_{ik} = (\rho + p)\, u_i u_k - p g_{ik} \ .$$

Formally $\mathcal{L}_m = p \equiv \frac{1}{2} \left((\rho + p) u_i u_k g^{ik} - (\rho - p) \right)$.

The Einstein equations with matter and a cosmological constant

$$S = -\frac{1}{16\pi G}\int (R + 2\Lambda)\sqrt{-g}\, d\Omega + \int \mathcal{L}_m \sqrt{-g}\, d\Omega \ .$$

$$\frac{\delta S}{\delta g^{ik}} = 0 \to R_{ik} - \frac{1}{2} g_{ik} R = g_{ik}\Lambda + 8\pi G T_{ik} \ , \ R = -4\Lambda - 8\pi G T \ .$$

Transition to the Newtonian limit:

$$g_{00} = 1 + 2\varphi, \ \ T_0^0 = \rho, \ \Gamma_{00}^{\alpha} = \frac{\partial \varphi}{\partial x^{\alpha}}, \ R_0^0 = -\frac{R}{2} = \frac{\partial \Gamma_{00}^{\alpha}}{\partial x^{\alpha}} = \triangle \varphi \ ,$$

$$\triangle \varphi = 4\pi G \rho$$