Lectures on relativistic gravity and cosmology.

Lectures 9-10

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Basics of cosmology

The FLRW cosmological model

The most important solutions of the model

Light propagation in the FLRW universe

Cosmology and the Universe

Subject of cosmology:

general properties of the present Universe at large scales, its past and future

Definition of the Universe.

Practical modern scientific definition of the Universe corresponding to the usage of this notion in physics and astronomy, taken from A. A. Starobinsky, *The Universe*, Physical Encyclopedia, vol. 1, pp. 346-348, Moscow, "Soviet Encyclopedia", 1988 (in Russian):

(The) UNIVERSE – all part of material world surrounding us and accessible to observations.

The definition is operational and avoids logical problems arising e.g. when using too much generic mathematical notions like 'set of all sets'.

Still now I prefer to extend this definition:

(The) UNIVERSE – all part of material world surrounding us and accessible to observations in the past, present and future.

Clearly, other universes are possible, too, at least logically. Continuing self-citation:

"Since the Universe does not necessarily exhaust by itself the whole objectively existing material world, the hypothesis about the existence of other universes is admissible. So far these universes are considered at a purely speculative level, they may be either always separated from our Universe, or may have a common origin with it from one primordial pre-universe. The latter possibility is realized e.g. in some variants of the inflationary universe scenario."

Characteristic scales

The nearest giant galaxy (similar to the Milky Way) - the Andromeda galaxy (M31): $L \approx 0.8$ Mpc $\approx 2 \cdot 10^{24}$ cm, $M \approx 10^{12} M_{\odot}$.
The nearest rich galaxy cluster - the Virgo cluster, $L \approx 17$ Mpc, $M \approx 10^{15} M_{\odot}$.
The last scattering surface (the surface of recombination) - $L = L_{hor} \approx 14000$ Mpc.

Primordial cosmic microwave background (CMB) radiation with the black-body (Planck) energy spectrum:

$$T_{\gamma} = (2.72548 \pm 0.00057)\,\mathrm{K}$$

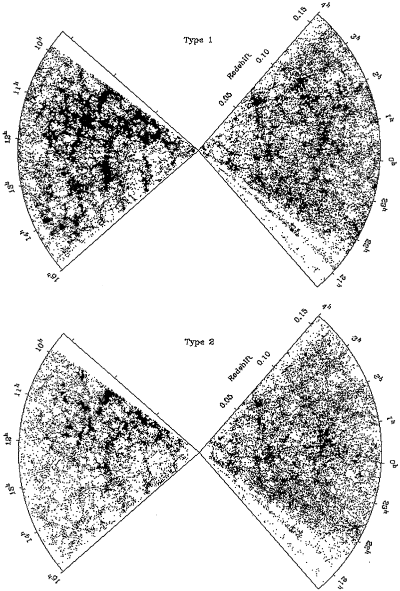

Figure 19. Redshift slices for different spectral types: type 1 corresponds to E/S0, type 2 to Sa/Sb, type 3 to Sc/Sd and type 4 to Irr.

The Hubble law and isotropy of the Universe

The Hubble law:

$$\mathbf{v} = H_0 \mathbf{r}, \quad H_0 = (70 \pm 3)\,\frac{\mathrm{km}}{\mathrm{s}\cdot\mathrm{Mpc}}, \quad H_0^{-1} \approx 14\,\mathrm{byl.\ y.}$$

with H_0 - isotropic. Valid for $r \gtrsim 3$ Mpc where r is the physical distance measured in the locally Minkowski reference frame and for $v = |\mathbf{v}| \ll 1$.

Other arguments for isotropy:
1) isotropic spatial distribution of galaxies and clusters, isotropy of galaxy counts;
2) optical sky isotropy: total light flux from galaxies is isotropic;
3) X-ray background isotropy;
4) CMB: T_γ is isotropic with accuracy $\sim 10^{-5}$ (after subtracting the dipole anisotropy).

Homogeneity of the Universe and the value of H_0

The Hubble law does not select the preferred origin of coordinates: its form with respect to any other point A in space which moves with the velocity $\mathbf{v}_A = H_0 \mathbf{r}_A$ is

$$\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{v}_A = H_0(\mathbf{r} - \mathbf{r}_A) = H_0 \tilde{\mathbf{r}}.$$

The unsolved problem of the value of H_0.
Cosmological tests based on the previous evolution of the Universe and GR (CMB and baryon acoustic oscillations (BAO)) give $H_0 = (67.7 \pm 0.4)\,\frac{\mathrm{km}}{\mathrm{s \cdot Mpc}}$ (arXiv:1807.06209).
Astronomical determination based on the combination of astronomical parallaxes, cepheids and supernova stars in the present Universe give $H_0 = (73.2 \pm 1.3)\,\frac{\mathrm{km}}{\mathrm{s \cdot Mpc}}$ (arXiv:2012.08534).
However, another astronomical determination based on red giant stars gives $H_0 = (69.6 \pm 1.3\,(\mathrm{stat}) \pm 1.7\,(\mathrm{sys}))\,\frac{\mathrm{km}}{\mathrm{s \cdot Mpc}}$ (arXiv:2002.01550).

Old and new cosmology

3 'whales' on which 'old' classic cosmology was based:

1. GR
Gravitational field is described by a space-time metric satisfying the Einstein equations $\mathcal{L} = -\frac{R}{16\pi G} + \mathcal{L}_m$

2. Approximate homogeneity and isotropy of the Universe.

3. Hot past (the Big Bang).

Change from 'old to 'new' standard cosmology.

1. Understanding that all 3 basic foundations are approximate.
a) There exist natural generalizations of the Einstein equations, and we need them.
b) The Universe may be and generically is indeed very anisotropic and inhomogeneous at very large scales $L \gg L_{hor}$ not observable now.
c) The very early Universe may still be cold.

2. Discovery of two new kinds of dark 'entities': dark matter and dark energy (cosmological constant, in particular).

3. Discovery of two new periods in the evolution of the Universe in the very remote past and at the present time when its expansion is accelerated.

In the present standard cosmology, there is no need to introduce the 'Copernican principle' (namely, that our position in space is typical) as an additional principle: it follows from equations and observational data.

The FLRW cosmological model

Three types of the isotropic homogeneous (Friedmann-Lemaître-Robertson-Walker) cosmological model: closed (positively spatially curved), flat and open (negatively spatially curved).

$$ds^2 = dt^2 - a^2(t)(dr^2 + \sin^2 r\, d\Omega^2),\ \mathcal{K} = 1,\ 0 \le r \le \pi,$$

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2\, d\Omega^2),\ \mathcal{K} = 0,\ 0 \le r < \infty,$$

$$ds^2 = dt^2 - a^2(t)(dr^2 + \sinh^2 r\, d\Omega^2),\ \mathcal{K} = -1,\ 0 \le r < \infty,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

For $\mathcal{K} = 1$, the total space volume is $V = 2\pi^2 a^3$.

Friedmann equation and the critical density

The $0-0$ Einstein equation with matter in the form of an ideal barotropic fluid having the equation of state $p = p(\rho)$ gives:

$$H^2 + \frac{\mathcal{K}}{a^2} = \frac{8\pi G\rho + \Lambda}{3}, \quad H = \frac{\dot{a}}{a}.$$

We shall call $H(t)$ the Hubble function, $H(t_0) = H_0$.

The energy-momentum tensor conservation

$\dot{\rho} + 3H(\rho + p) = 0$, or $d\mathcal{E} = -pdV$, can be easily integrated:

$$3 \ln a = -\int \frac{d\rho}{\rho + p(\rho)}.$$

By introducing the critical density

$\rho_c = \frac{3H_0^2}{8\pi G} \approx 0.92 \times 10^{-29} \left(\frac{H_0}{70}\right)^2$ g/cm^3 and $\Omega_i = \frac{\rho_i(t_0)}{\rho_c}$ for all types of matter and radiation, where t_0 is the present moment of time, the Friedmann equation for $t = t_0$ takes the form:

$$\sum_i \Omega_i + \Omega_{curv} + \Omega_\Lambda = 1, \ \Omega_{curv} = -\frac{\mathcal{K}}{a^2(t_0)H_0^2}, \ \Omega_\Lambda = \frac{\Lambda}{3H_0^2}.$$

Some simple and important solutions

The spatially flat case: $\mathcal{K} = 0$

I. One ideal fluid with the linear equation of state:
$p = \alpha\rho, \ 0 \leq \alpha \leq 1\,.$

$$a(t) \propto t^q\,, \quad q = \frac{2}{3(1+\alpha)}\,.$$

$\alpha = 0$ $\quad a(t) \propto t^{2/3}$ – matter-dominated stage;
$\alpha = 1/3$ $\quad a(t) \propto t^{1/2}$ – radiation-dominated stage.

The matter energy density

$$\rho = \frac{3H^2}{8\pi G} = \frac{3q^2}{8\pi G t^2}\,.$$

The main and dramatic consequences:

1. The Universe is not static, it is expanding.
2. The visible Universe with stars and galaxies is not eternal, it was completely different about the time period $\sim H_0^{-1}$ in the past. In particular, this explains the Olbers paradox.
3. There was a singularity ($\rho \to \infty$) in the past.
4. For $\alpha \geq 0$, the lifetime of the Universe from the singularity is too short: $T = \frac{2}{3H_0} \approx 9.3$ byl. years for $\alpha = 0$, while there exist old globular clusters with ages not less than 12 byl. years.

II. Dust-like matter + radiation:

$\rho = \rho_m + \rho_r, \quad p_m = 0, \quad p_r = \rho_r/3\,.$

$$a(\eta) = a_1\eta(\eta + \eta_1), \qquad \eta = \int \frac{dt}{a(t)},$$

$$t(\eta) = a_1\eta^2\left(\frac{\eta}{3} + \frac{\eta_1}{2}\right), \quad \eta_{eq} = \eta_1\,\frac{\sqrt{2}-1}{2} \approx 0.207\eta_1.$$

$\eta(t)$ is called conformal time.

III. Cosmological constant + spatial curvature = de Sitter space-time

$$R_{\alpha\beta\gamma\delta} = H_0^2(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta_\gamma})$$

4 most popular forms of its space-time metric (only the first metric covers the whole space-time):

$$ds^2 = dt_c^2 - H_0^{-2}\cosh^2(H_0 t_c)\,(d\chi_c^2 + \sin^2\chi_c d\Omega^2),$$

$$ds^2 = dt^2 - a_1^2 e^{2H_0 t}\,(dr^2 + r^2 d\Omega^2),\ \ a_1 = const,$$

$$ds^2 = dt_o^2 - H_0^{-2}\sinh^2(H_0 t_o)\,(d\chi_o^2 + \sinh^2\chi_o d\Omega^2),$$

$$ds^2 = (1 - H_0^2\tilde{r}^2)d\tau^2 - (1 - H_0^2\tilde{r}^2)^{-1}d\tilde{r}^2 - \tilde{r}^2 d\Omega^2.$$

IV. Vacuum + negative spatial curvature ($\mathcal{K} = -1$): the Milne metric $a(t) = |t|$.
A part of the flat space-time inside the future (for $t > 0$) or past ($t < 0$) light cone of some event.

V. Dust-like matter + positive cosmological constant – the standard ΛCDM model (neglecting radiation).

$$a(t) = a_1 \left(\sinh \frac{3H_1 t}{2} \right)^{2/3}, \ \Lambda = 3H_1^2 = 3H_0^2 \left(1 - \Omega_m\right).$$

Consequence for the quantity (called the first statefinder parameter, or jerk): $s = \frac{\dddot{a}\, a^2}{\dot{a}^3} \equiv 1\,.$

Home tasks

Find $a(t)$ for:

1) dust + radiation + spatial curvature,
2) radiation + cosmological constant,
3) the Einstein unstable static solution for dust + positive spatial curvature + cosmological constant.

Light propagation and redshift

$$ds = 0 \to r = \pm\eta\,.$$

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k} = 0 \to \psi = k(x-\eta),\ \omega = \frac{k}{a}\,.$$

For massive particles: $\mathbf{p}a(t) = const,\ \gamma v a(t) = const$ where p is the physical relativistic momentum.

The redshift $z(t)$ for light emitted in the past and observed at present is defined as $z = \frac{\omega(t)-\omega_0}{\omega(t)}$. Thus,

$$z = \frac{a(t_0)}{a(t)} - 1,\ \eta_0 - \eta = a_0^{-1}\int_0^z \frac{dz}{H(z)}\,.$$

The luminosity distance

The luminosity distance D_L is defined through $I = L/D_L^2$. On the other hand:

$$I = L\,\frac{a^2(\eta_0 - \eta)}{a^4(\eta_0) r^2}\,,\;\; r = \eta_0 - \eta\,.$$

$$D_L = a_0(\eta_0 - \eta)(1+z) = (1+z)\int_0^z \frac{dz}{H(z)}\,.$$

For $z \ll 1$ and restoring c, $D_L(z) = cz/H_0$.

Inversion of the formula gives the model-independent determination of $H(z)$ from $D_L(z)$:

$$H(z) = c\left[\frac{d}{dz}\left(\frac{D_L(z)}{1+z}\right)\right]^{-1}\,.$$

However, the practical application of this formula requires smoothing over some interval of redshift.