FRIEDMANN COSMOLOGY IN ALTERNATIVE DIMENSIONS

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Introduction

If one breaks up a habit to consider integer dimensionalities, he can open new possibilities for at least theoretical studying our Universe. A popular subject is considering multidimensional models with integer dimensions. However, we think that dimensionality may be not only noninteger, but also variable at different scales: It may change with relative distances between bodies. Moreover, we prefer to speak not of extra but of lacking dimensionalities. The dimensionality which is equal (or approximately equal) to 3 at the laboratory scales should be less than 3 at larger scales. It seems that there exist some convincing arguments. First, passing to greater and greater scales, from galaxies to clusters and superclusters of galaxies, the discrepancy between the luminous matter and its dynamics becomes stronger and stronger. To avoid this difficulty, one usually introduces the concept of dark matter that is required in increasing amounts when passing to larger scales [1] or try to modify the Newtonian dynamics [2]. Second, there exist the discrepancy between the age of the galaxies [3] and globular clusters [4] and that of the relatively small age of our Universe in the standard cosmology [5]. These discrepancies can be resolved if one recalculates the above dynamics and age for the case of a space dimensionality less than 3.

However, in refusing of the integer and constant spatial dimensionality, we can do the same with the temporal dimensionality because general relativity learns us that space and time should be considered at equal grounds. We have no consistent theory to describe our propositions in full although there exists an approach to general relativity involving noninteger dimensionalities [6], but we construct here Friedmann-like cosmological models which are in principle solvable due to their simplicity and symmetry. Our results have some preliminary meaning. We present also traditional cosmological tests for arbitrary spatial dimensions.

Homogeneous and Isotropic World in n Spatial Dimensions

Consider the metric interval in the form

$$ds^2 = c^2\,dt^2 - a^2(t)[dr^2 + \tilde{\phi}^2(r)\,d\sigma^2_{(n)}] \tag{1}$$

which corresponds to the Riemannian space which is a topological product of the time axis and an isotropic n-dimensional space. The quantity $d\sigma_{(n)}$ is the angle-distance element at the surface of n-dimensional Euclidean sphere, the but it is not required explicitly. The function $\tilde{\phi}(r)$ is the same as in the 3-dimensional case [5]:

$$\tilde{\phi}(r) = \begin{cases} \sin\,r, & 0 \le r \le \pi, \quad k = +1; \\ r, & r \ge 0, \quad k = 0; \\ sh\,r, & r \ge 0, \quad k = -1. \end{cases}$$

The parameter $k=-1,0,+1$ corresponds to the sign of the Gauss curvature.

Let the space-time be filled by the hydrodynamic matter with the stress-energy tensor

$$T_0^0=\epsilon, \quad T_0^i=0, \quad T_i^k=-p\delta_i^k, \tag{2}$$

$(i,k,...=1,2,...n)$, note that the trace of (2) now is $T=\epsilon-np$.

A standard way of calculating the Einstein equations gives (a dot denotes differentiation with respect to ct):

$$\frac{1}{2}n(n-1)\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)=\frac{\kappa^{(n)}}{c^2}\epsilon, \tag{3}$$

$$(n-1)\left[\frac{\ddot{a}}{a}+\frac{n-2}{2}\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)\right]=-\frac{\kappa^{(n)}}{c^2}p, \tag{4}$$

and from the conservation law $T^{\nu}_{\mu;\nu}=0$ $(\mu,\nu,...=0,1,...,n)$ it follows that

$$\dot{\epsilon}+n\frac{\dot{a}}{a}(\epsilon+p)=0. \tag{5}$$

We also present the equation with the "eliminated curvature"which is the combination of Eqs. (3) and (4)

$$n(n-1)\frac{\ddot{a}}{a}=-\frac{\kappa^{(n)}}{c^2}[(n-2)\epsilon+np]. \tag{6}$$

The equation of state is taken in the habitual form

$$p=q\epsilon \tag{7}$$

with q some parameter: $q=0$ for the dust matter and $q=1/n$ for pure radiation. We assume $q\neq-1$ in this Section because $q=-1$ is an isolated case.

From here, we can forget that n is an integer and supply it with real values $n>2$. In eliminating the quantities ϵ and p from Eqs. (3), (5) and (6), we obtain the general solution for the scale factor $a(t)$ in the implicit form when $k=-1$ and $k=+1$

$$\frac{2}{n(q+1)}A\left(\frac{a}{A}\right)^{\frac{n(q+1)}{2}}{}_2F_1\left(\frac{1}{2},\gamma;\gamma+1;k\left(\frac{a}{A}\right)^{n(q+1)-2}\right)=ct \tag{8}$$

where ${}_2F_1$ is the Gauss hypergeometric function, A is some integration constant,

$$\gamma=\frac{n(q+1)}{2[n(q+1)-2]}.$$

For $k=+1$, the solution (8) gives a branch which increases until the maximal value of the scale factor A: $a_{max}=A$; the time of reaching a_{max} is

$$t_{max}^{(n)}=\frac{2\sqrt{\pi}}{n(q+1)}\cdot\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)}\,\frac{A}{c}, \tag{9}$$

here and below Γ is the gamma function. The periodic solution can be composed from matching together the branches similar to the above and similar to those obtained by an operation of reflecting the above with respect to the line $t=t_{max}^{(n)}=Const$.

The asymptotics of the solution (8) for small t is given by the formula

$$a \propto t^{\frac{2}{n(q+1)}}. \tag{10}$$

The flat solution $k = 0$ is represented explicitly by the same formula (10) for the whole time axis.

We should remark that, first, the expressions (8), (9) and (10) are reducible to the same known expressions for the case $n = 3$. Second, it is interesting to note that the parameter n appears in the above formulae in the only combination $n(q + 1)$. Thus, the latter plays the role of an effective spatial dimensionality: for different but suitably chosen n and q the solution can be the same. For example, the dust solution in $n + 1$ dimensions is equivalent to the pure-radiation solution in n dimensions. Third, the restriction $n > 2$ is caused by the fact that for $q = 0$ and $k = +1$ an analytic limit of (8) when n tends to 2 does not exist, however this limit exists for $k = -1$.

As mentioned, the case $n = 2$ is distinct. It is also interesting by that there exists the stationary solution in the dust case ($q = 0$) where the scale factor is connected with the matter energy density:

$$a = c\,(\kappa^{(2)}\epsilon)^{-1/2}. \tag{11}$$

It is worth recalling that the Einstein equations without a cosmological constant in $n \neq 2$ dimensions have no physically acceptable stationary solutions. Besides (11), the linear solutions ($a \propto t$) take place for all k in the 2-dimensional dust case.

In order to derive cosmological tests in the next section, we introduce the dimensionless density parameter

$$\Omega^{(n)} = \frac{\epsilon}{\epsilon_c}$$

where the critical density ϵ_c is determined as usual by settling k equal to zero in Eq.(3):

$$\epsilon_c = \frac{n(n-1)}{2}\,\frac{H^2}{\kappa^{(n)}}$$

with $H = c\dot{a}/a$ the Hubble parameter. The deceleration parameter $q^{(n)}$ can be defined as usual and expressed via $\Omega^{(n)}$ with the help of Eq.(6):

$$q^{(n)} \equiv -\frac{c^2\ddot{a}}{aH^2} = \frac{n(q+1)-2}{2}\,\Omega^{(n)}.$$

Cosmological Tests in n Spatial Dimensions and the Age of the Universe

We present here the three traditional cosmological tests [7], in $n = 3$ these tests were derived in [5]: visual magnitude, angular size and the number of sources vs. redshift derived for the dust-filled Universe. We keep the only two orders in the redshift z; although the exact expressions for the tests have different analytic forms for different values of k, the above two orders surprisingly coincide analytically for all n like in the known case $n = 3$.

Visual magnitude vs. redshift. This test is very sensitive to the possible change of the dimensionality because it depends on the volume of the region between a source and an observer who measures the luminosity. Assume conditionally that the dimensionality is

3 if the distance between any bodies is less than R_0 and it is n otherwise. If the ratio $R_0 H_0/cz$ is small for small z, then the distance modulus $m - M$ is

$$m - M = 5z \left[\frac{5-n}{4} - \frac{(n-1)(n-2)}{8} \right] lg\, e - 5+$$

$$2.5 \left[(n-1) lg \frac{cz}{H_0} + (3-n) lg \frac{R_0}{\sqrt{2}} - lg \beta^{(n)} \right] + \frac{5}{16} \frac{(n-1)(3-n)}{n+1} \frac{R_0 H_0}{cz}$$

where, apart from R_0, the subscript "0"means that a given quantity is taken at the modern epoch, $\beta^{(n)}$ is some constant:

$$\beta^{(n)} = \frac{2^{(n+1)/2}\, \Gamma(n/2)}{(n-1)\sqrt{\pi}\, \Gamma((n-1)/2)}.$$

Angular size vs. redshift. The ratio $\tilde{R}$ of the source linear size l to its angular size θ can be expressed as follows:

$$\tilde{R} \equiv \frac{l}{\theta} = \frac{c}{H_0} \left[z - \left(\frac{3}{2} + \frac{n-2}{4} \Omega_0^{(n)} \right) z^2 \right] = \frac{c}{H_0} \left[z - \frac{1}{2} (3 + \bar{q}_0^{(n)}) z^2 \right]$$

where $\bar{q}_0^{(n)} = (n-2)\Omega_0^{(n)}/2$ is the deceleration parameter for the dust-filled Universe. This test is insensitive to a possible change of dimensionality inside a given z because we suppose that light rays propagate uniformly in spaces with any dimensionalities. Unfortunately, in the first two orders in z the contributions of n and $\Omega_0^{(n)}$ are undistinguishable. We can also see that the expression of $\tilde{R}$ via $\bar{q}_0^{(n)}$ is formally independent of n.

The number of sources vs. redshift. The differential of the number of sources at a given z is

$$dN = \nu_0\, s^{(n)} \left(\frac{c}{H_0} \right)^n z^{n-1} \left[1 - \frac{n+1}{2} (1 + q_0^{(n)}) z \right]$$

where ν_0 is the density of the sources, its physical dimensionality is (length)$^{-n}$, $s^{(n)}$ is the area of an n-dimensional sphere:

$$s^{(n)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

This test could be the most powerful one in order to determine n unless it was mostly contaminated by evolution and selection effects.

We should add that the presented tests hardly make easier the tasks for observers because they involve one or more additional parameters: n and/or R_0. The above tests evidently reduce to familiar ones in the case $n = 3$.

The age of the Universe is given by the following integral which cannot be expressed via elementary functions except of the cases $n = 2, 3$

$$T^{(n)} = H_0^{-1} \int dz\, (1+z)^{-2} [1 - \Omega_0^{(n)} + \Omega_0^{(n)} (1+z)^{n(q+1)-2}]^{-1/2}.$$

For $n = 3$ the resulting expression is in agreement with an expression in [5]. We can see that for n varying from 3 to 2, $T^{(n)}$ increases from some value $T^{(3)} < H_0^{-1}$ to exactly H_0^{-1}: $T^{(2)} = H_0^{-1}$ independently of the value of $\Omega_0^{(n)}$. For $k = 0$ ($\Omega_0^{(n)} = 1$), $T^{(n)} = 2H_0^{-1}/n(q+1)$.

Homogeneous and Isotropic World in n Spatial Dimensions and m Time Dimensions

Let now the Riemannian space be the topological product of m-dimensional flat "temporal"space and n-dimensional isotropic space (m and n are integers as yet). Let the metric interval have the form [8]

$$ds^2 = c^2\,(dt_1^2 + dt_2^2 + \,...\, + dt_m^2) - a^2(t)[dr^2 + \tilde{\phi}^2(r)\,d\sigma^2_{(n)}]$$

where the radial coordinate in the temporal space has the meaning of time:

$$t = (t_1^2 + t_2^2 + \,...\, t_m^2)^{1/2}. \tag{12}$$

As before, let $i, k, ... = 1, ..., n$ and let the capital Latin letters be referred to the temporal space: $A, B... = 1, ..., m$.

In order to make our cosmological model free of possible restrictions we have to introduce the new parameter p' which describes a matter tension in the temporal space leading to the stress-energy tensor written in the form

$$T_A^B = \epsilon\, u_A\, u^B - p'(\delta_A^B - u_A\, u^B), \quad T_A^i = 0, \quad T_i^k = -p\, \delta_i^k. \tag{13}$$

Clearly, the parameter p' does enter T_A^B when $m = 1$. The vector $u_A = t_A/t$ is the unit vector field in the temporal space. Moreover, the quantities $u_A = t_A/t$ represent the m components of the $(m+n)$-dimensional vector field which has the meaning of the $(m+n)$ velocity. As usual, its spatial components are zero: $u_i = 0$. In the case $m = 1$, it is the habitual 4-velocity vector.

Now we can calculate the generalized Einstein equations by introducing some coupling constant $\kappa^{(m,n)}$ and taking (13) as the source. We obtain the following set of equations where a dot denotes differentiation with respect to ct with the quantity (12) taking as the time:

$$n\left[\frac{m-1}{ct}\,\frac{\dot a}{a} + \frac{n-1}{2}\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)\right] = \frac{\kappa^{(m,n)}}{c^2}\,\epsilon, \tag{14}$$

$$n\left[\frac{\ddot a}{a} + \frac{m-2}{ct}\,\frac{\dot a}{a} + \frac{n-1}{2}\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)\right] = -\frac{\kappa^{(m,n)}}{c^2}\,p', \tag{15}$$

$$(n-1)\left[\frac{\ddot a}{a} + \frac{m-1}{ct}\,\frac{\dot a}{a} + \frac{n-2}{2}\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)\right] = -\frac{\kappa^{(m,n)}}{c^2}\,p. \tag{16}$$

The conservation law $T^\nu_{\mu;\nu} = 0$ ($\mu, \nu, ... = A, B, ... \oplus i, k, ...$) is, as usual, the integrability condition of Eqs. (14)-(16). It has now the form

$$\dot\epsilon + \frac{m-1}{ct}\,(\epsilon + p') + n\frac{\dot a}{a}\,(\epsilon + p) = 0.$$

As before, from now we consider n and m not integers but real numbers: $n > 1$ and $m > 0$. In order to make the set of Eqs. (14)-(16) solvable, we employ the equation of state (7). Then, in multiplying (14) on q and taking the sum with (16) using (7), we obtain the equation for the scale factor:

$$\frac{\ddot a}{a} + \frac{m-1}{m_{cr}-1}\,\frac{1}{ct}\,\frac{\dot a}{a} + \frac{1}{2}\left(\frac{m_{cr}-(2-n)}{m_{cr}-1} - 1\right)\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\right) = 0 \tag{17}$$

where we have denoted by

$$m_{cr} = 2 - \frac{nq}{n(q+1)-1}$$

some critical value of the temporal dimensionality. After that, Eq.(15) will determine the parameter p' after substituting there a solution $a(t)$ found from Eq.(17). Unfortunately, Eq.(17) has no general solutions in elementary functions. We present here the only solutions for some physically interesting cases.

1) *The flat n-dimensional space ($k=0$).* In this case Eq.(17) with $k=0$ is reducible to an equation in full differentials. Consider the following subcases:

1a) $m \neq m_{cr}$ and $q \neq -1$. The solution is

$$a = C\, t^{\gamma_{1a}}, \qquad \gamma_{1a} = \frac{2(m_{cr}-m)}{m_{cr}-(2-n)} \tag{18}$$

where C is an arbitrary number. Let $n > 1$ and $q > -(n-1)/n$, then $m_{cr} > (2-n)$ and the behavior of (18) is mainly determined by the sign of $m_{cr} - m$. If $m < m_{cr}$, then the scale factor increases from zero to infinity: The derivative $\dot{a}$ is zero at the point $t = 0$ for $m < [m_{cr} - (2-n)]/2$ and it is equal to infinity for $[m_{cr} - (2-n)]/2 < m < m_{cr}$. If $m > m_{cr}$, then the scale factor collapses from infinity to zero during infinite time. The time dependence of the matter parameters is as follows

$$\epsilon, p', p \propto t^{-2}. \tag{19}$$

1b) $m = m_{cr}$ and $q \neq -1$. The solution is logarithmic and starts from the point $t = 1$:

$$a = C\,(lnt)^{\gamma_{1b}}, \qquad \gamma_{1b} = \frac{2}{n(q+1)}.$$

In this case

$$\epsilon, p', p \propto (t\, lnt)^{-2}. \tag{20}$$

Now we turn to the case $q = -1$. It is interesting by its physical content giving rise to inflationary solutions [9] as well as by that it is an isolated case for Eq.(17). The solutions below contain the dependence on n only via m_{cr}, $m_{cr} = 2 - n$ here.

1c) $m \neq m_{cr}$ and $q = -1$

$$a = C_1\, exp(C_2\, t^{\gamma_{1c}}), \qquad \gamma_{1c} = \frac{m_{cr}-m}{m_{cr}-1} = \frac{m-(2-n)}{n-1}. \tag{21}$$

Due to that we have assumed before $n > 1$, $m_{cr} < 1$ and the behavior of (21) is mainly determined by the signs of $m_{cr} - m$ and the constant C_2. If $m > m_{cr}$, then all the solutions start at the point $t = 0$, $a = C_1$; for $C_2 > 0$ a solution infinitely increases and $C_2 < 0$ a solution decreases up to zero. In this case, for $m_{cr} < m < 1$ and $m > 1$ all the solutions start from $t = 0$ with the infinite derivative $\dot{a}$ and the zero derivative, respectively. If $m < m_{cr}$, the stationary regime $a = C_1$ is reached for infinite time and the above derivative at $t = 0$ has always infinite modulus: For $C_2 > 0$ a solution decreases from infinity and for $C_2 < 0$ a solution increases from zero. As to the matter parameters,

$$\epsilon, p', p \propto t^{2(\gamma_{1c}-1)}.$$

1d) $m = m_{cr}$ and $q = -1$. In this simple subcase the solution is linear: $a = Ct$, and the time dependence of the matter parameters is again (19).

2) *The stationary case.* This case generalizes the $n = 2$ stationary solution (11). Here, any dependence on m disappears, if $a = Const$, then

$$\epsilon = \frac{n(n-1)}{2} \frac{k}{a^2}, \quad p' = -\epsilon = -\frac{n(n-1)}{2} \frac{k}{a^2}, \quad p = -\frac{(n-1)(n-2)}{2} \frac{k}{a^2}.$$

meaning that $q = -(n-2)/n$. For $n > 2$, the stationary Universe cannot be filled by ordinary matter: it must possess negative energy density or negative pressure.

3) *The 2-dimensional ($n = 2$) dust-filled ($q = 0$) spacetime.* It is interesting that the solution is the same for all k:

$$a = \begin{cases} C\, t^{2-m}, & m \neq 2; \\ C\, ln\, t, & m = 2. \end{cases}$$

Note that here $m_{cr} = 2$ and the time dependence of the matter parameters is the same as in (19) and (20) for $m \neq 2$ and $m = 2$, respectively.

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