NONSTANDARD VARIATIONAL PRINCIPLES IN QUANTUM MECHANICS WITH SELF-INTERACTION

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Abstract

A nonlinear quantum mechanics with self-consistent interaction is constructed by a way which provides some features of classical field theory, but does not preclude from probability interpretation. The main feature of this theory is its invariance under the rescaling transformations of wave functions. Some fictitious, unobtainable exactly, Lagrangian function and the variational principles of its smoothed extremality, instead of those of extremality of an action, are the necessary elements of this construction.

1 Introduction

The standard self-consistent way of including interaction in classical field theories, based on the localization of symmetries, is well-known, it was introduced long ago [1] and generalized for gravitation in [2]. However this way is not directly suitable for theories which one wants to interpret as quantum-mechanical. Quantum mechanics (first quantization) remains up date linear theory, without included interaction, since one believes that the superposition principle provides probability interpretation. Quantum theories with interaction are just the second quantized field theories. Recall that the procedure of second quantization differs from that of first quantization, the former operates with the field space. However, the main drawback of quantum field theories is the appearance of divergencies, often immovable. Constructions presented here give an alternative way for inclusion of quantum (self-) interaction which perhaps could resolve the problems of second quantized theories.

Thus, the inherent feature of quantum mechanics is its probability interpretation. This means that the three postulates of the probability theory must be fulfilled i) there exists a unit-valued functional making the sense of full probability, ii) the full probability is the sum of individual probabilities for mutually exclusive events, and iii) the full probability is the product of individual probabilities for independent events. These postulates are easily incorporated in linear quantum mechanics. Indeed, the (linear) rescaling transformation for the wave function Ψ,

$$\Psi \rightarrow a\Psi \tag{1.1}$$

with a an arbitrary complex number, permits one to supply with unit value a conserved norm-square functional. The second postulate is realized for the superposition of orthogonal states, and the third one is suitable for the standard many-particle problem.

The main problem of the inclusion of (self-)interaction into a theory which one wants to interpret by a quantum-mechanical way is to preserve the invariance of the theory under (1.1). In the paper [3], we outlined the program of such the inclusion for gravitational self-interaction into one-particle quantum mechanics. (We use the term "self-interaction" for one particle.) It was continued in [4] and accomplished in [5, 6] for the gravitational interaction. The inclusion of electromagnetic interaction was considered in [7]. An approach to the many-particle problem of quantum mechanics was also described [7, 5, 6]. We already discussed interpretation features [3, 5] and argued for that all the postulates of the probability theory can also survive in the framework of the nonlinear theory with interaction.

Without claims to a mathematical rigor, we give here a general recipe of constructing a theory invariant under (1.1) which gives just equations slightly different from those in a classical field theory. The evolution equation for a wave function remains formally the same. In equations for an interaction field, a source term is a "classical" source divided by the conserved norm-square, that permits one to interpret the source as some probability density. However, this is achieved by introducing exotic variational principles: the principles of extremality of an action are replaced by some principles of smoothed extremality. We make our generalizations step by step, paying attention to intermediate stages and general facts. Below, we review them following the plan of the paper. Some preliminary mathematics for this is adduced in [8, 9, 10].

In Section 2, we consider an evolution equation which is taken with an arbitrary order of time derivatives, and the nature of the interaction field (or fields) is not specified. It is important to note (especially if we deal with the inclusion of gravity [5, 6]) that wave functions should be 3-dimensional scalar densities. We also assume that wave functions are always square-integrable over any required 3-domain. We present the functionals of a norm-square, a bilinear action and a bilinear current, which are associated with the evolution equation. We assume that the norm-square functional is positive definite. Certainly, the problem of nonpositive-definite norm-square functionals appears in theories with higher time derivatives [11] but it has nothing to do with the inclusion of self-interaction.

In Section 3, we represent a remarkable mathematical fact which, to our knowledge, is not described earlier. We introduce a formal differentiation with respect to the operator of time differentiation The fact is that, for any bilinear action with arbitrary operators and of an arbitrary order n of time derivatives, without reference to any especially chosen wave functions (and, moreover, to any solutions of the evolution equation), the derivative of the bilinear action with respect to the time-differentiation operator coincides with the norm-square functional of Section 2. This result is the most important in what follows. It gets a realization in terms of stationary states in Section 4 and in terms of so-called time-separated states in Section 5. It also lays the foundation of the constructions of Sec.7 and Sec.8.

In Section 4, we recall the Rayleigh variational principle. It was introduced in [12]. This is just the principle for obtaining an eigenvalue problem. Its degenerate form appears in the standard Schrodinger quantum mechanics [13]. Further development of the Rayleigh formalism one can find in the monograph [14]. We already gave the description the Rayleigh formalism [3, 4] and here we also account it in brief.

Section 5 is devoted to the case when the evolution equation admits the separation of time and space variables, although we do not formulate conditions of such the separation. The separation of space variables for typical equations of mathematical physics was considered in [15], some algorithm was constructed for obtaining a system of coordinates where the separation can be done. We outline a problem which generalizes the eigenvalue problem polynomial in a spectral parameter and introduce the concepts of a time-separated state and a basic function which generalize those of an eigenfunction and a spectral parameter, respectively. Generalization of the Rayleigh formalism is also given. Some solvable examples are considered. On one hand, Section 5 represents only a preparatory stage for inclusion of self-interaction because special form of time-separated states precludes from it. On the other, it may be of interest by itself.

In Section 6, we recall the standard procedure of inclusion of interaction which is suitable for classical field theories and write standard variational principles.

Section 7 is a central section of this paper. We present here some formal constructions, just the variational principles of smoothed extremality leading to a theory with desired features.

In Section 8, we give our formulation of the many-particle problem of quantum mechanics. It was not self-consistently formulated so far [16]. Traditionally it was restricted by the non-relativistic motion of particles, and, moreover, self-interaction of particles was excluded. We consider M quantum particles, no matter of various or the same sorts, described by a wave function which is the product of M individual functions Ψ_I ($I = 1, 2, \ldots, M$) and extend our variational principles to the M-particle problem. Formally, our equations resembles equations of a classical field theory which describes M different matter fields with a distinction similar to above. However, our theory remain invariant under the rescaling transformations

$$\Psi_I \to a_I \Psi_I \tag{1.2}$$

for all I. The inclusion of interaction between particles exposes the problem of a relation between the usual 3-dimensional space and the $3M$-dimensional space and the impossibility to include interaction in the only 3-space [17]. This problem is discussed.

2 Evolution Equation

Let the wave function $\Psi(t, x)$ satisfy the evolution equation of the order n

$$\sum_{k=0}^{n} \hat{A}_k \, \partial^k \Psi = 0 \tag{2.1}$$

where $\partial \equiv i\partial/\partial t$, $\hat{A}_k$ are some formal differential operators which do not already contain time differentiation but contain the differentiation with respect to the 3-space coordinates x. Let $\hat{A}_k$ also involve some sufficiently smooth real functions $\gamma(t, x)$ which will be treated as physical fields,

$$\hat{A}_k = \hat{A}_k[\gamma(t, x)].$$

Note that the presence of γ supplies $\hat{A}_k$ with a time dependence. We settle below that, in any expression, ∂ and $\hat{A}_k$ act until an unclosed right round bracket or, otherwise, on the whole right penetrating any closed brackets. The equation complex conjugate to Eq.(2.1) is

$$\sum_{k=0}^{n} \hat{A}_k^* \, \partial^{*k} \Psi^* = 0. \tag{2.2}$$

As for the hermiticity properties of $\hat{A}_k$, the transposed ($\hat{A}_k^T$) and complex conjugate ($\hat{A}_k^*$) operators ought to be connected by the relation:

$$\sum_{k=0}^{n} \hat{A}_k^* \, \partial^{*k} \ \ldots = \sum_{k=0}^{n} \partial^{*k} \hat{A}_k^T \ \ldots \tag{2.3}$$

which follows from the equality

$$\hat{A}_k^* = \sum_{l=k}^{n} C_l^k (\partial^{*(l-k)} \hat{A}_l^T)$$

where $C_l^k = l!/[k!(l-k)!]$ is the binomial coefficient. Clearly, the operators $\hat{A}_k$ are generally non-Hermitean: $\hat{A}_k^* \neq \hat{A}_k^T$; the hermiticity is restored for time-independent operators.

We associate with Eq.(2.1) the following functional which plays the role of a Lagrangian function bilinear in Ψ^* and Ψ:

$$K(t) = K[\Psi^*, \Psi] = \sum_{k=0}^{n} \int d^3x \; \Psi^* \; \hat{A}_k \; \partial^k \Psi \equiv \sum_{k=0}^{n} < \Psi^* \cdot \hat{A}_k \; \partial^k \Psi >, \tag{2.4}$$

the complex conjugate functional associated with Eq.(2.2) being

$$K^*(t) = K^*[\Psi^*, \Psi] = \sum_{k=0}^{n} \int d^3x \; (\hat{A}_k^* \; \partial^{*k} \Psi^*) \; \Psi \equiv \sum_{k=0}^{n} < \hat{A}_k^* \; \partial^{*k} \Psi^* \cdot \Psi > .$$

Using the relation (2.3), one can see that

$$K(t) - K^*(t) = \partial N^2(t) \tag{2.5}$$

where the functional N^2 can be expressed in several equivalent forms, and here we give the two of them:

$$N^2(t) = N^2[\Psi^*, \Psi] = \sum_{k=0}^{n} \sum_{l=0}^{k-1} < \partial^{*l} \hat{A}_k^T \; \Psi^* \cdot \partial^{k-l-1} \Psi >$$

$$= \sum_{k=1}^{n} \sum_{l=0}^{k-1} (-1)^l C_k^{l+1} \partial^l < \hat{A}_k^T \Psi^* \cdot \partial^{k-l-1} \Psi > . \tag{2.6}$$

Therefore, under the condition $K = 0$ (hence, $K^* = 0$) which is the weaker condition than the fulfillment of Eq.(2.1), the conservation law

$$\partial N^2 = 0 \tag{2.7}$$

holds. The functional (2.6) makes the sense of a norm-square functional and gives the standard Schrodinger norm square in the case of the first order Schrodinger equation. Although it is not immediately seen from (2.6), $N^2 = (N^2)^*$. Note that (2.7) is valid for any γ independently of its origin, thus (2.6) is also suitable for the case of self-interaction.

We can introduce the bilinear action functional which is associated with the Lagrangian (2.4):

$$J = \int dt \; K(t) \doteq \int dt \; K^*(t) \tag{2.8}$$

where the dot equality denote neglecting of some initial and final terms, [cf. (2.5)], which have no significance for functional differentiation.

We also define the bilinear current functional which corresponds to the field γ:

$$Q^{bil}(t, x) = Q^{bil}[\Psi^*, \Psi] = \frac{\delta J}{\delta \gamma(t, x)}. \tag{2.9}$$

Note that Q^{bil} is a real quantity due to the reality of γ.

3 Differentiation with Respect to the Operator of Time Differentiation

In the previous section, we introduced functionals which contain the operator of time differentiation expressed as ∂ or ∂^* depending on what function it is applied. Here, we give formal rules of differentiation with respect to ∂ and ∂^* for the functionals of the type (2.4). One can construct analogous rules by a less formal way if to express the operator of time differentiation via a time derivative of the δ-function with a difference of time arguments and to express the functionals of the type (2.8) via contractions of these functions. This was done in [8]. However, the rules presented here are more simple and clear.

Thus, define the derivatives with respect to ∂ and ∂^* which we denote by the letters Δ in the symbol of differentiation (Δ/Δ):

$$\frac{\Delta}{\Delta\partial} < \Theta^* \cdot \partial\Lambda >= \frac{\Delta}{\Delta\partial} < \partial^*\Theta^* \cdot \Lambda >=< \Theta^* \cdot \Lambda >, \tag{3.1}$$

$$\frac{\Delta}{\Delta\partial^*} < \Theta^* \cdot \partial\Lambda >= \frac{\Delta}{\Delta\partial^*} < \partial^*\Theta^* \cdot \Lambda >=< \Theta^* \cdot \Lambda > . \tag{3.2}$$

It can be easily shown that these rules imply that the functional derivatives with respect to ∂ and ∂^* of a total time derivative vanish. Indeed,

$$\frac{\Delta}{\Delta\partial}\partial < \Theta^* \cdot \Lambda >= \frac{\Delta}{\Delta\partial}(- < \partial^*\Theta^* \cdot \Lambda > + < \Theta^* \cdot \partial\Lambda >) = 0.$$

The rules (3.1) and (3.2) should be completed by the rule of differentiation of arbitrary-order time derivatives, that is by some generalization of the standard Leibnitz rule:

$$\frac{\Delta}{\Delta\partial} < \Theta^* \cdot \partial^k\Lambda >=< \Theta^* \cdot \frac{\Delta}{\Delta\partial}\partial\partial^{k-1}\Lambda > + \ldots + < \partial^{*(p-1)}\Theta^* \cdot \frac{\Delta}{\Delta\partial}\partial\partial^{(k-p)}\Lambda >$$
$$+ \ldots + < \partial^{*(k-1)}\Theta^* \cdot \frac{\Delta}{\Delta\partial}\partial\Lambda >=< \Theta^* \cdot \partial^{(k-1)}\Lambda > + \ldots + < \partial^{*(p-1)}\Theta^* \cdot \partial^{(k-p)}\Lambda >$$
$$+ \ldots + < \partial^{*(k-1)}\Theta^* \cdot \Lambda >= \sum_{l=0}^{k-1} < \partial^{*l}\Theta^* \cdot \partial^{(k-l-1)}\Lambda > \tag{3.3}$$

where only the symbol ∂ which is the nearest to the operator of differentiation is immediately differentiated. The rules analogous to (3.3) for $\Delta/\Delta\partial^*$ and for another distribution of ∂^k between Θ and Λ can be restored without difficulty. The rule is clear: when ∂ [∂^*] at the p-th place is differentiated, all the remaining $(p-1)$ ∂ [∂^*] on the left are translated to the left [right] function Θ [Λ] with acquiring [losing] asterisks. It follows that the result of the above differentiation is independent of a concrete distribution of ∂ between Θ and Λ, the only summarized order of differentiation, k, is in essence.

Now let us differentiate the Lagrangian functional (2.4) with respect to ∂ and ∂^*, it is easy to verify that

$$\frac{\Delta}{\Delta\partial}K[\Psi^*, \Psi] = \frac{\Delta}{\Delta\partial^*}K[\Psi^*, \Psi] \equiv K' = N^2[\Psi^*, \Psi]. \tag{3.4}$$

Thus, the equality (3.4) contains an achievement of this section:

3.1°: *The functional derivative of the Lagrangian functional* (2.4) *with respect to the time differentiation operator is equal to the norm-square functional* (2.6).

In the next sections, we demonstrate that, for functions of some special types, the operator of time differentiation is replaced by equivalent multiplication operators, and that the proposition 3.1° always has an equivalent counterpart.

4 Stationary Case and Rayleigh Variational Principles

Let the operators $\hat{A}_k$ be independent of time and involve functions $\gamma(x)$, then they can be assumed Hermitean: $\hat{A}_k^* = \hat{A}_k^T$. Consider the stationary wave function

$$\Psi_s(t,x) = \exp(-i\lambda t)\psi(x), \tag{4.1}$$

where the spectral parameter λ is assumed to be a real number, and $\psi(x)$ is an eigenfunction to be determined. After substituting (4.1) into (2.1), the latter reduces to the eigenvalue problem with the nonlinear dependence on the spectral parameter λ (see also [11]):

$$\sum_{k=0}^{n} \lambda^k \hat{A}_k \, \psi(x) = 0. \tag{4.2}$$

The Lagrangian functional (2.4) on (4.1) is associated with Eq.(4.2):

$$K(\lambda) = K[\Psi_s^*, \Psi_s] = K[\psi^*, \psi; \lambda] = \sum_{k=0}^{n} \lambda^k < \psi^* \cdot \hat{A}_k \, \psi > . \tag{4.3}$$

We can calculate the derivative

$$\frac{\partial K(\lambda)}{\partial \lambda} \equiv K'(\lambda) = \sum_{k=1}^{n} k\lambda^{k-1} < \psi^* \cdot \hat{A}_k \, \psi >, \tag{4.4}$$

leading to the proposition

4.1°. *The functional* (4.4) *coincides with the norm-square functional* (2.6) *on the function* (4.1):

$$K'(\lambda) = N^2[\Psi_s^*, \Psi_s] = N^2[\psi^*, \psi; \lambda].$$

The stationary bilinear γ-current (2.9) on (4.1) can be rewritten as follows

$$Q^{bil}[\Psi_s^*, \Psi_s] = Q^{bil}[\psi^*, \psi; \lambda] = \frac{\delta^{(3)} K(\lambda)}{\delta^{(3)}\gamma(x)} = \sum_{k=0}^{n} \lambda^k \frac{\delta^{(3)}}{\delta^{(3)}\gamma(x)} < \psi^* \cdot \hat{A}_k[\gamma] \, \psi > . \tag{4.5}$$

The index "(3)" in functional derivatives means that it is applied to functionals performed by integration over the coordinates x [4, 9].

Let the functional R_s be some real solution to the equation

$$K(R_s) = K[\psi^*, \psi; R_s] = \sum_{k=0}^{n} (R_s)^k < \psi^* \cdot \hat{A}_k \, \psi >= 0 \tag{4.6}$$

where the functional $K(R_s)$ has the same form as (4.3); R_s is called the Rayleigh functional. Eq.(4.6) is an algebraic equation of the order n with real coefficients due to the hermiticity of $\hat{A}_k$. Let also the condition

$$K'(R_s) \neq 0 \tag{4.7}$$

be fulfilled. Then:

4.2°. *The extremum condition of R_s with respect to $\psi^*(x)$:*

$$\frac{\delta^{(3)} R_s}{\delta^{(3)} \psi^*(x)} = 0,$$

is equivalent to Eq.(4.2) *in terms of* R_s:

$$\sum_{k=0}^{n} (R_s)^k \hat{A}_k \, \psi(x) = 0.$$

Indeed, the rule of differentiation of a complete function gives

$$\frac{\delta^{(3)} R_s}{\delta^{(3)} \psi^*(x)} = -\frac{\delta^{(3)} K(R_s)}{\delta^{(3)} \psi^*(x)} \Big/ K'(R_s) = -\left(\sum_{k=0}^{n} (R_s)^k \hat{A}_k \, \psi(x) \right) \Big/ N^2[\psi^*, \psi; R_s]. \tag{4.8}$$

The set of extremum values of some R_s as a given solution to (4.6) must coincide with some subset of eigenvalues, see [14, 4] for details, and the set of functions which supply this R_s with extrema must coincide with the corresponding subset of eigenfunctions. The set of extrema of all the solutions R_s covers the whole set of eigenvalues. The condition (4.7) in terms of λ: $K'(\lambda) \neq 0$, guarantees the absence of associated vectors in the space of eigenfunctions [18, 11].

Define the Rayleigh γ-current:

$$Q_s = -\frac{\delta^{(3)} R_s}{\delta^{(3)} \gamma(x)}. \tag{4.9}$$

then:

4.3°. *The Rayleigh* γ*-current* (4.9) *is equal to the bilinear* γ*-current* (4.5) *divided by the norm-square functional, both of them expressed in terms of* R_s,

$$Q_s = Q^{bil}[\psi^*, \psi; R_s] \Big/ N^2[\psi^*, \psi; R_s].$$

The proof is based on a formula similar to (4.8).

We note that, first, the extremum condition of real R_s with respect to ψ gives a correct equation for the function ψ^*, and that, second, the difference between the currents (4.5) and (4.9) is the only denominator with $N^2[\psi^*, \psi; R_s]$. The latter fact provides the rescaling invariance of (4.9) under the transformations $\psi \to a\psi$ and gives a possibility to include into a theory stationary self-interaction. For the case of gravity, it was done in [3].

5 Time-Separated States and Generalized Rayleigh Variational Principles

Consider a wave function which possesses the specific property

$$\partial \Psi_{ts}(t, x) = \chi(t) \, \Psi_{ts}(t, x), \tag{5.1}$$

and which we call a time-separated (ts-)state; χ, which is in general a complex function of time, being called a basic function.

The integration of (5.1) shows that, for this function, a time variable is separated from space variables, that justifies its name,

$$\Psi_{ts}(t, x) = \exp[-i \int dt \, \chi(t)] \, \psi(x)$$

where ψ is an arbitrary function of x. We assume henceforth that the operators in Eq.(2.1) are such that they admit the separation of t and x, although there are no necessity to know any special form of these operators.

In applying the time derivative of the order k to Ψ, we can write

$$\partial^k \Psi_{ts}(t,x) = ([\chi(t)+\partial]^k\, 1)\; \Psi_{ts}(t,x) \tag{5.2}$$

where the expression in the parentheses on the right-hand side of (5.2) is merely a multiplier which represents the k-fold application of the operator $[\chi+\partial]^k$ to unity. We further denote

$$\mathcal{D}_k\chi = [\chi+\partial]^k\, 1. \tag{5.3}$$

For $k = 1, 2, 3, \ldots$ we have

$$\mathcal{D}_0\chi = [\chi+\partial]^0\, 1 = 1, \quad \mathcal{D}_1\chi = [\chi+\partial]^1\, 1 = \chi, \quad \mathcal{D}_2\chi = [\chi+\partial]^2\, 1 = \chi^2 + \partial\chi,$$

$$\mathcal{D}_3\chi = [\chi+\partial]^3\, 1 = \chi^3 + 3\chi\,\partial\chi + \partial^2\chi, \quad \ldots \quad .$$

The operators A_k are commutative with $\chi(t)$ and their time derivatives because they involve the only differentiation with respect to x ($l = 0, 1, 2, \ldots$):

$$[A_k, \partial^l \chi] = 0, \tag{5.4}$$

The substitution of (5.3) into (2.1) and the use of (5.4) lead to the equation

$$\sum_{k=0}^{n} \mathcal{D}_k\chi\, \hat{A}_k\, \Psi_{ts} = 0. \tag{5.5}$$

The stationary function (4.1) is certainly a partial case of a ts-state. However, despite that the problem of finding the ts-states and basic functions is some generalization of the eigenvalue problem we can hardly expect any continuity between basic functions and spectral parameters, because, unlike Eq.(4.2), Eq.(5.5) involve the derivatives of χ of the order $n-1$. The exclusion is the degenerate case of the generalized Schrodinger theory ($n = 1$) where the basic function can be treated as a time-dependent spectral parameter.

Now, we give an example of time separation in a second order theory ($n = 2$). Consider the Klein–Gordon equation with the electromagnetic interaction

$$[\partial - e\Phi]^2\Psi - \sum_{\alpha=1}^{3}\left[i\frac{\partial}{\partial x^\alpha} + eA_\alpha\right]^2 \Psi - m^2\Psi = 0.$$

In order to admit time-separation, let the scalar potential Φ be the only function of time, $\Phi = \Phi(t)$, and let the 3-vector potential A_α ($\alpha = 1,2,3$) be the only function of the 3-space coordinates x, $A_\alpha = A_\alpha(x)$. Let $\psi_{\mathbf{k}}(x)$ be an eigenfunction of the generalized momentum square operator:

$$\sum_{\alpha=1}^{3}\left(i\frac{\partial}{\partial x^\alpha} + eA_\alpha\right)^2 \psi_{\mathbf{k}}(x) = \mathbf{k}^2\psi_{\mathbf{k}}(x)$$

with $\mathbf{k}^2$ an eigenvalue (suitable boundary conditions are implied), then the quantity χ, which now acquires the index $\mathbf{k}$, is a solution to the equation

$$\partial(\chi_{\mathbf{k}} - e\Phi) + (\chi_{\mathbf{k}} - e\Phi)^2 - (\mathbf{k}^2 + m^2) = 0. \tag{5.6}$$

Eq.(5.6) is easily integrable and has the two solutions with the opposite signs:

$$\chi_{\mathbf{k}} = e\Phi(t) \pm (\mathbf{k}^2 + m^2)^{1/2}\, \frac{1 + \exp[\pm 2(\mathbf{k}^2 + m^2)^{1/2} t + C]}{1 - \exp[\pm 2(\mathbf{k}^2 + m^2)^{1/2} t + C]} \tag{5.7}$$

where C is a constant. There also exist the isolated solutions

$$\chi_{\mathbf{k}} = e\Phi(t) \pm (\mathbf{k}^2 + m^2)^{1/2}. \tag{5.8}$$

The solutions (5.8) can be obtained from (5.7) by tending C to minus infinity: $C \to -\infty$. If Φ is independent of time, $\Phi = Const$, then $\chi_{\mathbf{k}}$ coincides with the spectral parameter $\lambda_{\mathbf{k}}$, which formally have the same value as (5.8).

The Lagrangian functional (2.4) on a ts-state can be associated with Eq.(5.5):

$$K[\chi(t)] = \sum_{k=0}^{n} \mathcal{D}_k \chi < \Psi^*_{ts} \cdot \hat{A}_k \, \Psi_{ts} > . \tag{5.9}$$

We can calculate the following functional derivative

$$\frac{\delta}{\delta\chi(x)} \int dt' K[\chi(t')] \equiv K'[\chi(t)]$$

$$= \sum_{k=1}^{n} \sum_{l=0}^{k-1} (-1)^l C_k^{l+1} \partial^l [\mathcal{D}_{k-l-1} \chi(t) < \Psi^*_{ts} \cdot \hat{A}_k \, \Psi_{ts} >] \tag{5.10}$$

which leads to the proposition analogous to 4.1°.

5.1°. *The functional* (5.10) *coincides with the norm-square functional* (2.6) *on the ts-state*:

$$K'[\chi(t)] = N^2[\Psi^*_{ts}, \Psi_{ts}].$$

The γ-current (2.9) on the ts-state acquires the form

$$Q^{bil}[\Psi^*_{ts}, \Psi_{ts}] = \frac{\delta}{\delta\gamma(t,x)} \int dt' K[\chi(t')] =$$

$$\int dt' \sum_{k=0}^{n} \mathcal{D}_k \chi(t') \frac{\delta}{\delta\gamma(t,x)} < \Psi^*_{ts} \cdot \hat{A}_k[\gamma] \, \Psi_{ts} > . \tag{5.11}$$

Now, we construct the generalization of the Rayleigh functionals and principles. Let $R_{ts}(t)$ be some nonzero solution to the differential equation:

$$K[R_{ts}(t)] = \sum_{k=0}^{n} \mathcal{D}_k R_{ts} < \Psi^*_{ts} \cdot \hat{A}_k \, \Psi_{ts} >= 0, \tag{5.12}$$

and let the condition analogous to (4.7) be fulfilled:

$$K'[R_{ts}(t)] \neq 0.$$

Then:

5.2°. *The condition*

$$\left(\int dt' dt'' \frac{\delta K(t'')}{\delta R_{ts}(t')} \cdot \frac{\delta R_{ts}(t')}{\delta \Psi^*_{ts}(t,x)} \right) / K'[R_{ts}(t)] = 0, \tag{5.13}$$

is equivalent to Eq.(5.5) *in terms of* R_s

$$\sum_{k=0}^{n} \mathcal{D}_k R_{ts} \, \hat{A}_k \, \Psi_{ts} = 0.$$

The proof is based of the functional identity

$$\int dt' \frac{\delta K(t')}{\delta \Psi^*_{ts}(t,x)} + \int dt'dt'' \frac{\delta K(t'')}{\delta R_{ts}(t')} \cdot \frac{\delta R_{ts}(t')}{\delta \Psi^*_{ts}(t,x)} = 0 \tag{5.14}$$

where the differentiation in the first term is meant as if R_{ts} were independent of Ψ_{ts}.

We call the functional R_{ts} the generalized Rayleigh functional and the condition (5.13) the time-smoothed extremum condition of R_{ts} with respect to Ψ_{ts}. It is worth noting that the time-smoothed extrema of R_{ts} with respect to Ψ_{ts} are not the set of discrete numerical values as for the true Rayleigh functionals but some set of time-dependent functions.

The generalized Rayleigh current functionals should be defined with the use of time smoothing

$$Q_{ts} = -\left(\int dt'dt'' \frac{\delta K(t'')}{\delta R_{ts}(t')} \cdot \frac{\delta R_{ts}(t')}{\delta \gamma^*(t,x)} \right) \Big/ K'[R_{ts}(t)]. \tag{5.15}$$

5.3°. *The generalized Rayleigh γ-current* (5.15) *is equal to the bilinear current functional* (2.9) *divided by the norm-square functional* (2.6)*:*

$$Q_{ts} = Q^{bil}[\Psi^*_{ts}, \Psi_{ts}] \Big/ N^2[\Psi^*_{ts}, \Psi_{ts}].$$

The generalized Rayleigh functional can play the role of some Lagrangian function. We note, however, that the variational principle for obtaining an evolution equation is "one-sided", i.e., the time-smoothed extremum of the functional R_{ts} with respect to the function Ψ_{ts} does not give a correct evolution equation for Ψ^*_{ts}. This is because the time derivatives of Ψ_{ts} replaced by factors involving R_{ts} in the functional (5.12), cannot be converted into the time derivatives of Ψ^*_{ts} when integrating over t' in the first term of the identity (5.14). The exclusion is the case of Hermitean operators $\hat{A}_k(t)$ together with the real-valued functional $R_{ts}(t)$.

Now we return to Eq.(5.12) which is the nonlinear differential equation of the order $n-1$ and of some special kind (for $n = 2$ it is a partial Riccati equation, see below). The function R_{ts} as the solution to (5.12) can hardly be found explicitly, however, the principle of smoothed extremality and the definition of the generalized Rayleigh current are formulated without reference to any explicit solutions of these equations. Eq.(5.12) can be transformed into a linear equation of the order n using the transformation of variables $R_{ts}(t) = \partial \ln\, r(t)$, then

$$\mathcal{D}_k R_{ts}(t) = [R_{ts}(t) + \partial]^k\, 1 = \frac{\partial^k r}{r}.$$

After that Eq.(5.12) acquires the form

$$\sum_{k=0}^{n} < \Psi^*_{ts} \cdot \hat{A}_k\, \Psi_{ts} > \partial^k r = 0. \tag{5.16}$$

The linear equation (5.16) must have n independent solutions $r^{(i)}$, $i = 1, 2, \ldots, n$, so that its general solution is

$$r = \sum_{i=1}^{n} c_i r^{(i)}$$

with n arbitrary constants c_i. In terms of the variable R_{ts}, the solution has the form

$$R_{ts} = \partial \ln \sum_{i=1}^{n} c_i\, \exp\left(- \int dt\, R^{(i)}_{ts} \right).$$

In the case $n = 2$, Eq.(5.12) acquires the form

$$K(R) =< \Psi^*_{ts} \cdot \hat{A}_2 \, \Psi_{ts} > (R^2_{ts} + \partial R_{ts}) + < \Psi^*_{ts} \cdot \hat{A}_1 \, \Psi_{ts} > R_{ts} + < \Psi^*_{ts} \cdot \hat{A}_0 \, \Psi_{ts} >= 0$$

which is the partial Riccati equation, and which has no, in general, explicit solutions. However, there is a solvable example in the case $\hat{A}_0 = 0$, namely, the Bernoulli equation [19]:

$$a_2(t)(R^2_{ts} + \partial R_{ts}) + a_1(t) R_{ts} = 0 \tag{5.17}$$

where we have denoted

$$a_1(t) =< \Psi^*_{ts} \cdot \hat{A}_1 \, \Psi_{ts} >, \quad a_2(t) =< \Psi^*_{ts} \cdot \hat{A}_2 \, \Psi_{ts} > .$$

Eq.(5.17) has the explicit solution

$$R_{ts} = \partial \ln \left(-i \int dt \exp\{+i \int dt [a_1(t)]^{-1} a_2(t)\} \right) . \tag{5.18}$$

Obviously, the integrals in (5.18) are indefinite and thus the solution contains implicitly two integration constants. This example allows one to verify immediately the propositions 5.2° and 5.3°. The only point that should be stressed is that some accuracy should be done in calculating functional derivatives, see [10].

6 Standard Variational Principles and the Inclusion of Self-Interaction

For a classical gauge theory with self-interaction, we represent standard variational principles of extremality in a form which will be convenient below for a comparison with our modifications. Define the field γ as a gauge field which enters extended derivatives in order to make Eq.(2.1) invariant under a localized group of transformations suitable for the theory which is considered. Let $\Gamma \equiv Inv(\gamma)$ be a suitable invariant of the field γ. The procedure of such a kind was firstly introduced in [1]. For example, if we consider the group of localized phase transformations, then γ represents the electromagnetic four-potential [1]. For obtaining gravitation, the localization of Poincare transformations in the framework of the vierbein formalism requires the compensating connection field [2].

Consider the total action

$$J^{tot} = -\alpha^{-1} J^{int} + J = \int dt \, L^{tot}(t) \tag{6.1}$$

where α is an interaction constant,

$$J^{int} = \int d^3x' \, \Gamma(t, x')$$

and

$$L^{tot}(t) = -\alpha^{-1} \int d^3x' \, \Gamma(t, x') + K(t) \tag{6.2}$$

is the total Lagrangian function. Then:

6.1°. *The extremum condition of* (6.1) *with respect to* $\Psi^*(t, x)$ $[\Psi(t, x)]$*:*

$$\frac{\delta J^{tot}}{\delta \Psi^*(t, x)} = \int dt' \, \frac{\delta L^{tot}(t')}{\delta \Psi^*(t, x)} = 0$$

$$\left[\frac{\delta J^{tot}}{\delta\Psi(t,x)} = \int dt' \, \frac{\delta L^{tot}(t')}{\delta\Psi(t,x)}\right] = 0,$$

is equivalent to Eq.(2.1) [*Eq.*(2.2)].

6.2°. *The extremum condition of* (6.1) *with respect to* $\gamma(t,x)$*:*

$$\frac{\delta J^{tot}}{\delta\gamma(t,x)} = \int dt' \, \frac{\delta L^{tot}(t')}{\delta\gamma(t,x)} = 0,$$

is equivalent to the following interaction equation

$$\frac{\delta}{\delta\gamma(t,x)} \int dt' d^3x' \Gamma(t',x') = \alpha \, Q^{bil}(t,x) \tag{6.3}$$

where the $Q^{bil}(t,x)$ *is the bilinear current* (2.9).

It should be stressed that, first, although our consideration is based on the privileged role of time, the procedure of the inclusion of interaction in this Section is indeed relativistically invariant. In second, we couple replace K by K^* in (6.2), but this fact does not affect the propositions 6.1° and 6.2°. In this connection we note that Q^{bil} is a real quantity due to the reality of γ.

7 Another Version of Variational Principles with Self-interaction

Let us construct the functional

$$\mathcal{K}[t;\mathcal{R}] = \sum_{k=0}^{n} \int d^3x \, \Psi^*(\hat{A}_k \, \mathcal{D}_k\mathcal{R} \, \Psi) \tag{7.1}$$

which resembles the functional (5.9) but differs from it by some unspecified as yet quantity $\mathcal{R}$ which is a function of t and x: $\mathcal{R} = \mathcal{R}(t,x)$. The multiplier $\mathcal{D}_k\mathcal{R}$ is defined similar to (5.3)

$$\mathcal{D}_k\mathcal{R} = [\mathcal{R} + \partial]^k \, 1.$$

We shall need also the time-dependent functional of the type (5.10):

$$\int dt' \, d^3x \, \frac{\delta\mathcal{K}(t')}{\delta\mathcal{R}(t,x)} = \sum_{k=1}^{n}\sum_{l=0}^{k-1}(-1)^l \, C_k^{l+1} \, \partial^l \int d^3x \, (\hat{A}_k^T\Psi^*) \, \mathcal{D}_{k-l-1}\mathcal{R} \, \Psi = \mathcal{K}'. \tag{7.2}$$

Further, we require the condition

$$\partial\Psi = \mathcal{R} \, \Psi \tag{7.3}$$

which entails the two following preliminary propositions.

7.1°. *Under the condition* (7.3), *a) the functional* (7.1) *coincides with* (2.4), *b) the functional* (7.2) *coincides with* (2.6)*:*

$$\mathcal{K}'(t)|_{(7.3)} = N^2(t).$$

The validity of 7.1°a can be verified using the relation similar to (5.2)

$$\partial^k\Psi = \mathcal{D}_k\mathcal{R} \, \Psi \tag{7.4}$$

which is the consequence of (7.3); 7.1°b is evident after the comparison of (7.2) and (2.6) with the use of (7.4).

We should distinguish now local and nonlocal functionals. We call a functional $F[f]$ local if its functional derivative with respect to the function f contains no primitive functions of the δ-functions, otherwise this functional is called nonlocal. For example, $\mathcal{K}$ is the local functional of $\mathcal{R}$.

Henceforth, let $\mathcal{R}$ be some solution to the integro-differential equation

$$\mathcal{K}[t;\mathcal{R}] = 0, \tag{7.5}$$

where the left-hand side of (7.5) is defined by (7.1). Then $\mathcal{R}$ is, in general, a nonlocal functional of Ψ^*, Ψ and γ, hence it is some nonlocal function of t and x and may carry explicit dependences on t and x. However, the knowledge of the explicit form of $\mathcal{R}$ is unnecessary for further constructions.

7.2°. *If Eq.*(7.5) *is fulfilled simultaneously with the condition* (7.3), *then the conservation law* (2.7) *takes place for* (7.2): $\partial\mathcal{K}' = 0$.

This is true due to 7.1°b and the relation (2.5).

Now we start a new inclusion of self-interaction. Let us for a moment leave aside the condition (7.3). Instead of (6.2), consider another (nonlocal) Lagrangian function with K replaced by $-\hbar\mathcal{R}$:

$$L^{tot}(t,x) = -\alpha^{-1}\int d^3x'\,\Gamma(t,x') - \hbar\,\mathcal{R}(t,x). \tag{7.6}$$

(The Planck constant $\hbar$ is added to $\mathcal{R}$ to get the dimensionality of energy since the dimensionality of $\mathcal{R}$ is always inverse time independently of that of Ψ. Note here that the dimensionality of $\mathcal{K}$ depends on that of Ψ; by intention, the latter is chosen by such a way to absorb any dimensional factor by $\mathcal{K}$.)

Instead of the variational principles given by 6.1° and 6.2°, we construct the principles where the functional derivatives are replaced by some smoothed quantities. Consider the condition similar to that in Sec.5: Let the functional (7.2) be nonzero for any $\mathcal{R}$:

$$\mathcal{K}'(t) \neq 0. \tag{7.8}$$

For any functional $F[t,x;f]$ which can carry an explicit dependence on t and x, define the smoothed functional derivative with respect to f always keeping in mind (7.8) as follows:

$$\left(\frac{\delta F}{\delta f(t,x)}\right)_{sm} = \left(\int dt''\,dt'\,d^3x'\,\frac{\delta\mathcal{K}(t'')}{\delta\mathcal{R}(t',x')}\cdot\frac{\delta F(t',x')}{\delta f(t,x)}\right)\Big/\mathcal{K}'. \tag{7.9}$$

After the integration over the argument t'' the expression in the numerator of (7.9) can be represented in the form

$$\int dt'\,d^3x'\,\sum_{l=0}^{n-1}\left[\frac{\delta^{(3)}\mathcal{K}(t')}{\delta^{(3)}\partial'^{l}\mathcal{R}(t',x')}\partial'^{l}\right]\frac{\delta F(t',x')}{\delta f(t,x)}$$

where $\partial' \equiv i\partial/\partial t'$ and the index (3) as before indicates the "3-dimensional" functional derivative taken for fixed t'. This form is the most general in the sense that it is valid both for local and nonlocal functionals. The integrand of (7.9) is not generally the product of $\mathcal{K}(t')$ and the usual functional derivative of F with respect to f; this is the case for local functionals only. Equating (7.9) to zero can be called the smoothed extremum condition of F with respect to f.

The basic propositions are the following.

7.3°. *The smoothed extremum condition of* (7.6) *with respect to* $\Psi^*(t,x)$:

$$\left(\frac{\delta L^{tot}}{\delta \Psi^*(t,x)}\right)_{sm} = 0, \tag{7.10}$$

with the simultaneous fulfillment of the condition (7.3) *is equivalent to Eq.*(2.1).

To prove 7.3° note that the first term in (7.6) is independent of Ψ^* thus the smoothed extremum is taken from the second term. Substituting $\mathcal{R}$ which is a solution to (7.5) back to (7.5) and integrating the obtained identity over time, the usual functional differentiation of the result with respect to $\Psi^*(t,x)$ gives

$$\int dt' \frac{\delta \mathcal{K}(t')}{\delta \Psi^*(t,x)} + \int dt' dt'' d^3x' \frac{\delta \mathcal{K}(t'')}{\delta \mathcal{R}(t',x')} \cdot \frac{\delta \mathcal{R}(t',x')}{\delta \Psi^*(t,x)} = 0. \tag{7.11}$$

Dividing (7.11) by the nonzero functional (7.2), we can see that

$$\left(\frac{\delta \mathcal{R}}{\delta \Psi^*(t,x)}\right)_{sm} = -\left(\sum_{k=0}^{n} \hat{A}_k \, \mathcal{D}_k \mathcal{R} \, \Psi\right) / \mathcal{K}'$$

whence 7.3° becomes evident after using (7.3).

7.4°. *The smoothed extremum condition of* (7.6) *with respect to* $\gamma(t,x)$*:*

$$\left(\frac{\delta L^{tot}}{\delta \gamma(t,x)}\right)_{sm} = 0,$$

with the simultaneous fulfillment of the condition (7.3) *is equivalent to the following interaction equation*

$$\frac{\delta}{\delta \gamma(t,x)} \int dt' \, d^3x' \, \Gamma(t',x') = \hbar\alpha \, \frac{Q^{bil}(t,x)}{N^2}. \tag{7.12}$$

It is worth noting that the left-hand side of (7.12) is just that of (6.3), however, unlike (6.3) the bilinear current (2.9) on the right-hand side of (7.12) is divided by the conserved norm square.

Turn to the left-hand side of (7.12). When taking the smoothed functional derivative of the first term with respect to γ, due to the locality of Γ as the function of γ, these derivatives involve the only functions $\delta(t-t')$, $\delta(x-x')$ and their partial derivatives. That is why, in an expression of the type (7.10), the functional (7.2) which is independent of t' due to the proposition 7.2° can be extracted as the multiplier, put out the sign of an integral over t' and contracted with the same denominator, i.e.,

$$\left(\frac{\delta}{\delta \gamma(t,x)} \int dt' \, d^3x' \, \Gamma(t',x')\right)_{sm} = \frac{\delta}{\delta \gamma(t,x)} \int dt' \, d^3x' \, \Gamma(t',x').$$

As for the right-hand side of (7.12), consider the identity similar to (7.11)

$$\int dt' \frac{\delta \mathcal{K}(t')}{\delta \gamma(t,x)} + \int dt' dt'' d^3x' \frac{\delta \mathcal{K}(t'')}{\delta \mathcal{R}(t',x)} \cdot \frac{\delta \mathcal{R}(t',x')}{\delta \gamma(t,x)} = 0.$$

The first term of (7.13), after using the relation (7.4), the proposition 7.1°a and the fact that the operation of taking $\delta/\delta\gamma$ and that of substituting (7.4) commute, is the bilinear current (11). Therefore, recalling (7.8) , 3°b and 7.2°,

$$\left(\frac{\delta \mathcal{R}}{\delta \gamma(t,x)}\right)_{sm} = -\frac{Q^{bil}}{N^2}$$

and 7.4° becomes evident.

Propositions 7.3° and 7.4° show that we need not the notion of action but rather that of Lagrangian function. This is because the "smoothing functions" are disposed under the sign of time integration which usually makes action from a Lagrangian function.

Now, we have formulated the variational principles of smoothed extremality. We have achieved the desired result, namely, the invariance under the transformations (1.1). The only distinction from classical field theory is the appearance of the conserved norm-square functional which is, on one hand, a functional and, on the other hand, merely a number as a denominator on the right-hand side of interaction equations. However, this is done at the expense of losing the three important properties of the standard procedure, namely, (i) the principles of extremality, (ii) the non-privileged position of the time coordinate and (iii) the unambiguous way of constructing the Lagrangian function. The latter means that since $\mathcal{R}$ is a complex functional, we have the two possibilities: to choose $\mathcal{R}$ or $\mathcal{R}^*$ as the matter Lagrangian function. In the case of $\mathcal{R}^*$, Eq.(2.2) would be obtained as a smoothed extremum of $\mathcal{R}^*$ with respect to Ψ, whereas Eq.(7.12) is obtained as before: it is insensitive to the above choice. However, Eq.(2.2) [Eq.(2.1)] can never be obtained as the smoothed extremum of $\mathcal{R}$ [$\mathcal{R}^*$] with respect to Ψ [Ψ^*]. Thus our variational principles are "one-sided" by the same reason as in the Section 5, but this has no significance because one complex evolution equation is always enough for complete description of a physical situation.

Emphasize that the condition (7.3) is necessary to compensate introduced superfluous quantity $\mathcal{R}$, this condition generalizes an analogous condition for stationary states: $\partial\Psi_s(t,x) = \lambda\,\Psi_s(t,x)$, and the condition (5.1) for ts-states. It is sufficient to only use (7.3) in the smoothed extrema.

8 Many-Particle Problem and Inclusion of Interaction

Consider M particles each of which is described by the wave function $\Psi_I(t_I, x_I)$ where the index $I = 1, 2, \ldots, M$ shows that the function and its arguments belong to the I^{th} particle. Construct the wave function in the $(3+1)M$-space

$$\Psi(t_1, t_2, \ldots, t_M, x_1, x_2, \ldots, x_M) = \prod_{I=1}^{M} \Psi_I(t_I, x_I) \tag{8.1}$$

which is reduced to the usual wave function $\bar{\Psi}$ of the M-particle system defined in the $3M+1$-space for coinciding time arguments

$$\bar{\Psi}(t, x_1, x_2, \ldots, x_M) = \Psi(t, t, \ldots, t, x_1, x_2, \ldots, x_M). \tag{8.2}$$

Let each Ψ_I satisfy the equation of the type (2.1)

$$\sum_{k=0}^{n_I} \hat{A}_{Ik}(t_I, x_I)\, \partial_I^k \Psi_I(t_I, x_I) = 0 \tag{8.3}$$

where $\partial_I = i\partial/\partial t_I$, the equation's order n_I and the operators $\hat{A}_{Ik}$ are attached to the I^{th} particle, however, we suppose as before that these operators contain some field γ, the same for all the particles, but expressed as a function of the I^{th} arguments:

$$\hat{A}_{Ik} = \hat{A}_{Ik}[\gamma(t_I, x_I)].$$

As the consequence of (8.3), the function (8.1) satisfies the equation

$$\left[\prod_{I=1}^{M}\sum_{k=0}^{n_I}\hat{A}_{Ik}(t_I,x_I)\partial_I^k\right]\ \Psi(t_1,t_2,\ldots,t_M,x_1,x_2,\ldots,x_M)=0. \tag{8.4}$$

Define henceforth

$$d^{3M}x=d^3x_1\cdot d^3x_2\cdot\ldots\cdot d^3x_M,$$
$$d^Mt=dt_1\cdot dt_2\cdot\ldots\cdot dt_M.$$

For the Lagrangian function consider the following one suggested by Eq.(8.4) which has the dimensionality of (energy)^M:

$$K(t_1,t_2,\ldots,t_M)=\int d^{3M}x\ \Psi^*(t_1,t_2,\ldots,t_M,x_1,x_2,\ldots,x_M)\times$$

$$\times\left[\prod_{I=1}^{M}\sum_{k=0}^{n_I}\hat{A}_{Ik}(t_I,x_I)\ \partial_I^k\right]\ \Psi(t_1,t_2,\ldots,t_M,x_1,x_2,\ldots,x_M), \tag{8.5}$$

and which can certainly be decomposed to the product of M Lagrangian functions:

$$K(t_1,t_2,\ldots,t_M)=K_1(t_1)\ K_2(t_2)\ \ldots\ K_M(t_M) \tag{8.6}$$

where

$$K_I(t_I)=\sum_{k=0}^{n_I}\int d^3x\ \Psi_I^*(t_I,x)\ \hat{A}_{Ik}(t_I,x)\ \partial_I^k\Psi_I(t_I,x)\ .$$

Define also the functional which can be interpreted as the norm square functional for the function (8.1)

$$N^2(t_1,t_2,\ldots,t_M)=N_1^2(t_1)\ N_2^2(t_2)\ \ldots\ N_M^2(t_M) \tag{8.7}$$

where

$$N_I^2(t_I)=\sum_{k=1}^{n_I}\sum_{l=0}^{k-1}\int d^3x\ \left(\partial_I^{*l}\ \hat{A}_{Ik}^T(t_I,x)\ \Psi_I^*(t_I,x)\right)\ \partial_I^{k-l-1}\Psi_I(t_I,x) \tag{8.8}$$

is the norm square for the I^{th} particle [it is possible but rather cumbersome to express (8.7) like (8.5) via the function (8.1) without the above decomposition]. If we require $K_I=0$ for all I, then the conservation law for (8.8) of the type (2.7) holds. In this case, the functional (8.7) is conserved in each time argument:

$$\partial_I N^2(t_1,t_2,\ldots,t_M)=0.$$

The conserved functional (8.7) is a real number as the product of real numbers. We can also define the norm square functional for the function (8.2)

$$\bar{N}^2(t)=N^2(t,t,\ldots,t) \tag{8.9}$$

which is certainly conserved in t:

$$\partial\bar{N}=0. \tag{8.10}$$

Evidently, the conserved functionals (8.7) and (8.9) are equal numerically.

The definition (8.9) generalizes that in the usual M-particle Schrodinger theory. Such the definition reflects the third postulate of the probability theory: the probability of independent events is equal to the product of individual probabilities. This definition is also relevant for the system with interaction because the conservation law (8.10) is valid for any γ as before.

Using the Lagrangian function (8.5), it is possible to construct an M-particle bilinear action which is the product of M bilinear actions, however, any attempt to include an interaction term into such an action meets with a failure. This is because of the difficulties to bring in correlation the dimensionalities of matter and interaction terms in a total action and to get a sum of M sources in the interaction equation.

Similarly to previous constructions, consider the functional

$$\mathcal{K}[t_1, t_2, \ldots, t_M; \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_M] = \mathcal{K}_1[t_1; \mathcal{R}_1]\mathcal{K}_2[t_2; \mathcal{R}_2] \ldots \mathcal{K}_M[t_M; \mathcal{R}_M] \tag{8.11}$$

where

$$\mathcal{K}_1[t_I; \mathcal{R}_I] = \sum_{k=0}^{n_I} \int d^3x \; \Psi_I^*(t_I, x) \left[\hat{A}_{Ik}(t_I, x) \, \partial_I^k \Psi_I(t_I, x)\right],$$

and as before

$$\mathcal{D}_{IK} \, \mathcal{R}_I = [\mathcal{R}_I + \partial_I^k]1.$$

We also need the functional

$$\int d^Mt \, d^{3M}x \, \frac{\delta^M \, \mathcal{K}(t_1, t_2, \ldots, t_M)}{\prod_{I=1}^M \delta\mathcal{R}_I(t, x_I)} = \prod_{I=1}^M \int \, dt_I \, d^3x_I \, \frac{\delta\mathcal{K}_I(t_I)}{\delta\mathcal{R}_I(t, x_I)} \equiv \mathcal{K}'_M. \tag{8.12}$$

which depends on t. Note that all the $\mathcal{R}_I$ in (8.12) have the same time argument t which takes no part in the integration.

Let, as before, the following conditions be satisfied

$$\partial_I \Psi_I(t_I, x) = \mathcal{R}_I(t_I, x) \; \Psi_I(t_I, x) \tag{8.13}$$

for all I, then the following propositions extend 7.1°a, 7.1°b and 7.2° to the M-particle case.

8.1°. *Under the conditions* (8.13) *a) the functional* (8.11) *coincides with* (8.6), *b) the functional* (8.12) *coincides with* (8.7)*:*

$$\mathcal{K}'_M(t)|_{(8.13)} = \bar{N}^2(t).$$

Let each $\mathcal{R}_I$ be the solution to the equation

$$\mathcal{K}_I[t_I; \mathcal{R}_I] = 0, \tag{8.14}$$

as the consequence

$$\mathcal{K}[t_1, t_2, \ldots, t_M; \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_M] = 0.$$

Similarly to 7.2°,

8.2°. *If Eqs.*(8.14) *are fulfilled simultaneously with the conditions* (8.13)*, then the conservation law for* (8.12) *of the type* (8.10) *takes place:* $\partial\mathcal{K}'_M = 0$.

It worth noting that the conditions (8.13) yield the following condition for the function (8.2):

$$\partial\bar{\Psi}(t, x_1, x_2, \ldots, x_M) = \mathcal{R}(t, x_1, x_2, \ldots, x_M) \; \bar{\Psi}(t, x_1, x_2, \ldots, x_M) \tag{8.15}$$

where

$$\mathcal{R}(t, x_1, x_2, \ldots, x_M) = \mathcal{R}(t, x_1) + \mathcal{R}(t, x_2) + \ldots + \mathcal{R}(t, x_M). \tag{8.16}$$

Now, independently of the conditions (8.13) but taking (8.16) as the definition of $\mathcal{R}$, consider the nonlocal Lagrangian function

$$L^{tot}(t, x_1, x_2, \ldots, x_M) = -\alpha^{-1} \int d^3x' \, \Gamma(t, x') - \hbar \, \mathcal{R}(t, x_1, x_2, \ldots, x_M). \tag{8.17}$$

Let for all the $\mathcal{R}_I$ satisfying (8.14) the functional (8.12) be nonzero

$$\mathcal{K}'_M[t; \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_M] \neq 0. \tag{8.18}$$

For any functional $F[t, x_1, x_2, \ldots, x_M; f]$ which can carry an explicit dependences on t and x_I, we can define the M-smoothed (i.e., smoothed over M particles) functional derivative with respect to the function f, keeping in mind (8.18):

$$\left(\frac{\delta F}{\delta f(t,x)}\right)_{M-sm} = \left(\int dt' \, dt^M d^{3M}x \frac{\delta^M \mathcal{K}_I(..., t_I, ...)}{\prod_{I=1}^M \delta \mathcal{R}_I(t', x_I)} \cdot \frac{\delta F(t'..., x_I, ...)}{\delta f(t,x)}\right) / \mathcal{K}'_M. \tag{8.19}$$

Equating (8.19) to zero can be called the M-smoothed extremum condition of F with respect to f. Then, we have the following generalizations of the propositions 7.3° and 7.4°.

8.3°. *The M-smoothed extremum condition of* (8.17) *with respect to any* $\Psi_K^*(t,x)[(K = 1, 2, \ldots, M)]$*:*

$$\left(\frac{\delta L^{tot}}{\delta \Psi_K^*(t,x)}\right)_{M-sm} = 0,$$

with the simultaneous fulfillment of the conditions (8.13) *for all* $I = 1, 2, \ldots, M$ *is equivalent to the evolution equation:*

$$\sum_{k=0}^{n_K} \hat{A}_{Kk}(t,x) \, \partial^k \Psi_K(t,x) = 0.$$

Note that here we take the arguments t and x instead of t_I and x_I, cf. (8.3). Since $\Psi_K^*(t,x)$ enters only $\mathcal{R}_K$, we can rewrite the definition of the required M-smoothed derivative in the form:

$$\left(\int dt' \, dt_1 \, d^3x_1 \, \frac{\delta \mathcal{K}_1(t_1)}{\delta \mathcal{R}_1(t', x_1)} \, \ldots \, dt_K \, d^3x_K \, \frac{\delta \mathcal{K}_K(t_K)}{\delta \mathcal{R}_K(t', x_K)} \cdot \frac{\delta \mathcal{R}_K(t', x_K)}{\delta \Psi_K^*(t,x)} \, \ldots \times \right.$$

$$\left. \times \, dt_M \, d^3x_M \, \frac{\delta \mathcal{K}_M(t_M)}{\delta \mathcal{R}_M(t', x_M)}\right) / \prod_{I=1}^M \int dt_I \, d^3x_I \, \frac{\delta \mathcal{K}_I(t_I)}{\delta \mathcal{R}_I(t, x_I)}.$$

That all the factors $\delta\mathcal{K}_I/\delta\mathcal{R}_I$ integrated over t_I and x_I for $I \neq K$ are independent of t' due to 8.2° leads to the fact that they can be put out the sign of the integral over t' contracting with the same factors in the denominator. As the result, the M-smoothed extremum here is reduced to the "one-particle" smoothed extremum in 7.3°:

$$\left(\frac{\delta L^{tot}}{\delta \Psi_K^*(t,x)}\right)_{M-sm} = \left(\frac{\delta \mathcal{R}_K}{\delta \Psi_K^*(t,x)}\right)_{sm}.$$

Thus, 8.3° is valid as the consequence of 7.3° .

8.4°. *The M-smoothed extremum condition of* (8.17) *with respect to* $\gamma(t,x)$*:*

$$\left(\frac{\delta L^{tot}}{\delta \gamma(t,x)}\right)_{M-sm} = 0,$$

with the simultaneous fulfillment of the conditions (8.13) *for all* $I = 1, 2, \dots, M$ *is equivalent to the interaction equation:*

$$\frac{\delta}{\delta\gamma(t,x)} \int dt'\, d^3x'\, \Gamma(t',x') = \hbar\alpha \sum_{I=1}^{M} \frac{Q_I^{bil}(t,x)}{N_I^2}. \tag{8.20}$$

The proof of the validity of the left-hand side of (8.20) repeats mutatis mutandis that in 7.4°, in particular, one should refer to 8.2° instead of 7.2°. On the right-hand side of (8.20), by the same reason as in 8.3°, we have the sum of one-particle extrema,

$$\left(\frac{\delta\mathcal{R}}{\delta\gamma(t,x)}\right)_{M-sm} = \sum_{I=1}^{M} \left(\frac{\delta\mathcal{R}_I}{\delta\gamma(t,x)}\right)_{sm},$$

leading to the validity of 8.4°.

Thus, we have formulated the variational principles of M-smoothed extremality which reduce to those formulated for a single particle when $M = 1$. We stress that, just for the M-particle system, our way seems more natural and advantageous than any generalization of a "bilinear" way and leads to equations invariant under the rescaling transformation (1.2) applied to any wave function Ψ_I. Eq.(8.20) is very similar to a classical interaction equation for M matter fields, however the source contribution of each field on the right-hand side is divided by its own conserved norm square.

Concerning the mentioned relation between different spaces, we can see that the interaction indeed occurs in the $3+1$-space: this is guaranteed by the same arguments t and x for all the Ψ_K and γ. Nevertheless, the consistent formulation of our principles requires to deal with the $3M+1$-space at the level of the Lagrangian function and with the $(3+1)M$-space at the level of smoothing. We can imagine this by such a way that each I^{th} particle moves in its own I^{th} $(3+1)$-space, but these motions are not now independent and are determined by the interactions with other particles and the self-interaction in the common $(3+1)$-space. We have already shown how to come from our approach to the standard formulation of a many-particle problem with gravitational coupling by means of some inconsistent assumptions [6].

In conclusion, it is worth noting the succession between the conditions (7.3) and (8.15) both involving a usual wave function suitable for a given system, they always mean that

$$(wave\ function) \propto exp[-i \int dt\, \mathcal{R}(t)],$$

exiting associations with a quasiclassical wave function where, however, the exact Lagrangian function $-\hbar\mathcal{R}$ is taken instead of a classical one.

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