IS THE UNIVERSE ASYMPTOTICALLY SPATIALLY TWO DIMENSIONAL?

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1 Introduction

It was already argued [1, 2] that the dimensionality of our space may be expressed by a noninteger number and that it should be less than three on large scales in order to resolve the two cosmological problems. We should emphasize that we speak only about dimensionality depending on scale. Our everyday experience tells us that the dimensionality of our environment is three (or very near to it), and this is perhaps necessary in order for atoms to exist [3]. The same situation exists evidently in other galaxies at small scales, because we can see almost the same spectral lines, that indicate the same state of matter as here, and thus imply three dimensions. However, the situation at large scales can be drastically different: The dimensionality might change, and probably it smoothly decreases.

The first of the cosmological problems referred to is the contradiction between the dynamics of galaxies, their groups, clusters and superclusters to the amount of observed luminous matter: It is widely understood as being necessary that a much greater amount of dark matter should exist. For example, the dependence of mass on distance, $M(r)$, obtained from observing 21cm neutral hydrogen emission of gas clouds moving around a galaxy far from its visual bounds (the continuation of a rotation curve) [4, 5] is

$$M \propto r^{1.2 \div 1.3}. \tag{1.1}$$

This conclusion reflects the fact that the observed rotation velocity slightly increases at outer parts of galaxies, so the growth of M is interpreted as the presence of some dark halo besides luminous matter. Moreover, the amount of dark matter grows relatively to luminous matter when coming to larger and larger scales [6].

However, even the nonrelativistic (Newtonian) consideration of gravitational forces in spaces with lesser than three dimensions, done in Sec.2, enables us in principle to bring in correlation dynamics and "the shortage" of luminous matter. It would be very difficult to take into account the smooth fall of dimensionality because we do not know the law of such the fall. In order to make some estimates we roughly assume that on some relative distance R_0 the dimensionality changes by a leap from 3 to $n < 3$. We consider the rotation curves of disk galaxies, the dynamics of double galaxies and the dispersion of velocities in elliptical (spheroidal) galaxies. We show how one can lower the estimates of masses of these systems under the assumption that the dimensionality is lesser than three already at scales of the order of a typical galaxy's size. In Sec.2., we also discuss the possible hierarchical change of dimensionality. We only demonstrate the principal possibility to lower dynamical mass and refrain from making any numerical estimates as yet because of too many uncertain parameters.

The second cosmological problem is the discrepancy between the relatively small age of the Universe in the context of the standard (3-dimensional) Friedmann–Robertson–Walker (FRW) model on one hand and that following from the stellar evolution in galaxies

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and globular stellar clusters on the other. This problem had become of topical interest after recent measurements of the Hubble constant, which value is about 80 km/s/Mpc [7]. It was already shown that the age of the Universe in the context of n-dimensional FRW models is greater than that in the 3-dimensional FRW model if $n < 3$ [2]. We are inclined to believe that the space of our Universe is 2-dimensional in the limit of large cosmological scales. The reason is that the case $n = 2$ is in some sense a critical case for the Einstein equations: There exists a static dust solution without a cosmological constant. This is demonstrated in Sec.3., where we analyze the possibility of static n-dimensional solutions in the presence of the cosmological constant.

Certainly, we have no now a constructive physical model which can describe noninteger and nonconstant dimensionality. We can only outline some suggestive arguments for it. The first of possible suggestions comes from the fractal theory [8]; the space itself may have a fractal-like structure. The second suggestion is that effectively, if we consider individually each object in the Universe leaving aside other objects, then we can perceive a space between us and this object as 2-dimensional, because one spatial direction is fixed as a line from us to the object, and the other direction can be fixed by a vector of relative velocity of the object with respect to us. Thus, may be our space filled by distant separated objects consists of a set of (perhaps non-connected) 2-dimensional subspaces for which effectively $2 \leq n < 3$. The third suggestion comes from the existence of the large-scale structure of the Universe in distributions of galaxies, their groups, clusters, voids and superclusters. There is the tendency for matter to perform oblate structures at each hierarchical level. Possibly, the structure and dimensionality of our space itself might reflect the distribution of (luminous) cosmic matter.

2 Newtonian Consideration

In 3 and n dimensions, the expressions for the gravitational forces acting between the mass M and a unit mass separated by the distance r are

$$F^{(3)} = -\frac{G^{(3)}M}{r^2} \tag{2.1}$$

and

$$F^{(n)} = -\frac{G^{(n)}M}{r^{n-1}}, \tag{2.2}$$

respectively, where $G^{(3)}$ and $G^{(n)}$ are relevant gravitational constants. The corresponding potentials ($\vec{F} = \vec{\nabla}\Phi$ by definition) up to arbitrary constants $C^{(3)}$ and $C^{(n)}$ are

$$\Phi^{(3)} = \frac{G^{(3)}M}{r} + C^{(3)}, \tag{2.3}$$

and for $n \neq 2$

$$\Phi^{(n)} = \frac{1}{n-2}\,\frac{G^{(n)}M}{r^{n-2}} + C^{(n)}. \tag{2.4}$$

In the case $n = 2$,

$$\Phi^{(2)} = -G^{(2)}M\ln r + C^{(2)}. \tag{2.4a}$$

Now, let us assume that at some relative distance R_0 from a body the dimensionality changes by leap from 3 to n. The condition of matching the forces (2.1) and (2.2) at R_0 gives the connection between the gravitational constants

$$G^{(n)} = G^{(3)}R_0^{n-3}. \tag{2.5}$$

Thus, the improved force in accordance with our conception is (2.1) for $r \leq R_0$ and (2.2) for $r > R_0$ with (2.3):

$$F^{imp} = \begin{cases} F^{(3)}, & r \leq R_0, \\ F^{(n)}, & r > R_0. \end{cases} \tag{2.6}$$

The condition of matching the potentials (2.3) and (2.4) at R_0 is also required, and leads to the following expression for the "n-dimensional" constant $C^{(n)}$ [$C^{(3)} = 0$ is chosen in (2.3)],

$$C^{(n)} = -\frac{3-n}{n-2}\frac{G^{(3)}M}{R_0}$$

for $n \neq 2$, and

$$C^{(2)} = \frac{G^{(3)}M}{R_0}\,(1 + \ln R_0)$$

for $n = 2$.

Thus, the improved potential in accordance with our conception is (2.3) for $r \leq R_0$ and (2.4) [or (2.4a)] for $r > R_0$ with $C^{(n)}$ just defined:

$$\Phi^{imp} = \begin{cases} \Phi^{(3)}, & r \leq R_0, \\ \Phi^{(n)}, & r > R_0. \end{cases} \tag{2.7}$$

Let us stress that we can only think of R_0 as of a relative distance between any bodies. Otherwise, first, the conception of relativity of space which is an achievement of Einstein's physics, would be violated. Second, there would be troubles with the universality of a gravitational attraction.

Indeed, let a sphere with the radius R_0 be "rigidly fixed" in space, and let bodies freely come in it and come out it. Let a body of the mass M be placed inside the sphere at some point $\mathbf{O}'$ which does not coincide with the center of the sphere $\mathbf{O}$, and let two bodies of the equal small masses m be outside of this sphere at the points $\mathbf{A}_1$ and $\mathbf{A}_2$ in such a way that $\mathbf{O}'\mathbf{A}_1 = \mathbf{O}'\mathbf{A}_2 = r$. Then, we have for the gravitational forces between each body of the mass m and that of the mass M

$$F_1^{(n)} = -G_1^{(n)}\frac{mM}{r^{n-1}}, \qquad F_2^{(n)} = -G_2^{(n)}\frac{mM}{r^{n-1}}$$

where $G_1^{(n)} = G^{(3)}R_1^{n-3}$ and $G_2^{(n)} = G^{(3)}R_2^{n-3}$ are the gravitational constants found from the continuity of forces at the sphere, and R_1 and R_2 are the distances from the point $\mathbf{O}'$ to the points of intersections of the rays $\mathbf{O}'\mathbf{A}_1$ and $\mathbf{O}'\mathbf{A}_2$ with the sphere. Thus, if the above consideration were true, then the "gravitational constant" would not be a constant, and the law of gravitation attraction would be anisotropic and inhomogeneous. Moreover, in the case $n < 3$, the body of the mass M at $\mathbf{O}'$ which tends to the sphere on the inside ($R_1 \to 0$ or $R_2 \to 0$) would produce an infinitely divergent "gravitational constant".

That is why we admit R_0 as only relative distance. Perhaps, our consideration would be less rough, if we consider the change of dimensionality which occurs by leaps several times from 3 to n_1 at R_0, from n_1 to n_2 at R_1, and so on, and from n_j to $n_{j+1} \geq 2$ at R_j. Then, we have the chain of relations between the gravitational constants

$$\begin{aligned} G^{(n_1)} &= G^{(3)}R_0^{n_1-3}, \\ G^{(n_2)} &= G^{(n_1)}R_1^{n_2-n_1} = G^{(3)}R_0^{n_1-3}R_1^{n_2-n_1}, \\ &\dots \\ G^{(n_{j+1})} &= G^{(n_j)}R_j^{n_{j+1}-n_j} = \ldots = G^{(3)}R_0^{n_1-3}R_1^{n_2-n_1}\ldots R_j^{n_{j+1}-n_j}. \end{aligned} \tag{2.8}$$

The chain of relations between the constants in potentials is rather cumbersome; however the recurrence relation is $(i = 1, 2, ...j)$

$$C^{(n_{i+1})} = C^{(n_i)} - \frac{n_i - n_{i+1}}{(n_i - 2)(n_{i+1} - 2)} \frac{G^{(n_i)} M}{R_i^{n_i - 2}}.$$

The three subsections below present the application of the force (2.6) [or the potential (2.7)] to determinations of galactic masses. We shall return to discussing the problem of the dimensionality change after considering the three examples below.

2.1 Rotation Curves of Disk Galaxies

A rough calculation of rotation velocity v at the distance r far from the center of a galaxy (i.e., when all its mass is effectively concentrated near the center or spherically distributed around it) is based on the equality of the centrifugal force and gravitational force. In the 3-dimensional space, using (2.1)

$$\frac{v^2}{r} = \frac{G^{(3)} M_g^{(3)}}{r^2}. \tag{2.9}$$

In accordance with our conception, we can write

$$M_g^{dyn} \equiv M_g^{(3)} = \frac{v^2 r}{G^{(3)}},$$

i.e., we call the dynamical galactic mass, M_g^{dyn}, a mass calculated as if our space were 3-dimensional.

In the n-dimensional space, if $r \gg R_0$ then equality (2.9) should be replaced by the following [with the use of (2.2)]:

$$\frac{v^2}{r} = \frac{G^{(n)} M_g^{(n)}}{r^{n-1}},$$

so that we call a mass calculated in the n-dimensional space the true galactic mass, M_g^{true}:

$$M_g^{true} \equiv M_g^{(n)} = \frac{v^2 r^{n-2}}{G^{(n)}} = \left(\frac{R_0}{r}\right)^{3-n} M_g^{dyn} \tag{2.10}$$

with (2.5) used in the last equality. When $2 < n < 3$ the factor at M_g^{dyn} is lesser than unity, therefore $M_g^{true} < M_g^{dyn}$.

The more accurate calculation of the rotation curve can be done for the case when the disk of a galaxy lies in the 2-dimensional space (or plane) and there exist no other spatial dimensions. Let the distribution of the 2-dimensional matter density in the disk satisfy the law

$$\rho = \rho_0 \exp\left(\frac{r}{R_d}\right) \tag{2.11}$$

where ρ_0 is the 2-density in the disk center, and R_d is some characteristic radius. The distribution (2.11) corresponds to the observed distribution of luminous matter in disk galaxies [9]. Let $R_0 \ll R_d$. The velocity square in this case is given as follows

$$v^2(r) = 2\pi\rho_0 G^{(3)} \frac{R_d^2}{R_0}\left[1 - \left(1 + \frac{r}{R_d}\right)\exp\left(-\frac{r}{R_d}\right)\right]. \tag{2.12}$$

The function (2.12) monotonically increases from zero and tends to a constant value at $r \to \infty$. That is why the dynamical mass calculated with the aid of (2.12) tends to grow linearly at large r: $M_g^{dyn} \propto r$. However, at any finite r we can effectively write $M_g^{dyn} \propto r^\beta$ where always $\beta > 1$. Probably, this fact could explain the dependence (1.1), meaning that our real space has the effective dimensionality which is very near to two at the scales of the outer parts of galaxies.

2.2 Double Galaxy Systems

Let the two bodies (galaxies) **A** and **B** with the masses $m_{\mathbf{A}}$ and $m_{\mathbf{B}}$, respectively, rotate around a center of inertia. Let $\vec{r}_{\mathbf{A}}$ and $\vec{r}_{\mathbf{B}}$ be radii-vectors from an observer who is at rest relatively to the center of inertia of the above bodies, then $\vec{v}_{\mathbf{A}} \equiv d\vec{r}_{\mathbf{A}}/dt$ and $\vec{v}_{\mathbf{B}} \equiv d\vec{r}_{\mathbf{B}}/dt$ are the bodies' velocities with respect to the observer. Let the motion be circular for simplicity, then the distance $r \equiv |\vec{r}| = |\vec{r}_{\mathbf{A}} - \vec{r}_{\mathbf{B}}|$ between the bodies is constant.

First, consider the 3-dimensional case. For the motion reduced to the center of inertia, we have the equality of the centrifugal and gravitational forces:

$$\frac{m_{\mathbf{A}} m_{\mathbf{B}}}{m_{\mathbf{A}} + m_{\mathbf{B}}} \frac{v^2}{r} = G^{(3)} \frac{m_{\mathbf{A}} m_{\mathbf{B}}}{r^2} \tag{2.13}$$

where $v = |\vec{v}_{\mathbf{A}} - \vec{v}_{\mathbf{B}}|$ is the modulus of the relative velocity of the two bodies. From (2.13), we can infer the total dynamical ("3-dimensional") mass of the system

$$M_{tot}^{dyn} \equiv M_{tot}^{(3)} = m_{\mathbf{A}} + m_{\mathbf{B}} = \frac{v^2 r}{G^{(3)}}. \tag{2.14}$$

Let now the circular orbit and the observer's line of sight lie in the same plane, and let ϕ ($0 < \phi < \pi$) be an angle between the radius $\vec{r}$ and the line of sight. Then, the observer sees the projected distance between the bodies $r_p = r \sin\phi$ and measures the projected relative velocity $v_p = v \sin\phi$. That is why (2.14) should be rewritten in the form

$$M_{tot}^{dyn} = \frac{v_p^2 r_p}{G^{(3)} \sin^3\phi}, \tag{2.15}$$

which can be found, for example, in [10].

If we assume that in the n-dimensional space it is possible to construct the (2-dimensional) plane as before and that R_0 is not greater than a typical galaxy size, and hence $r > R_0$, then the expression (2.13) must be replaced by the following

$$\frac{m_{\mathbf{A}} m_{\mathbf{B}}}{m_{\mathbf{A}} + m_{\mathbf{B}}} \frac{v^2}{r} = G^{(n)} \frac{m_{\mathbf{A}} m_{\mathbf{B}}}{r^{n-1}},$$

whence the total true ("n-dimensional") mass is

$$M_{tot}^{true} \equiv M_{tot}^{(n)} = \frac{v^2 r^{n-2}}{G^{(n)}} = M_{tot}^{dyn} \left(\frac{R_0}{r_p}\right)^{3-n} (\sin\phi)^{3-n}. \tag{2.16}$$

For $2 < n < 3$, the product of the two factors on the right-hand side of (2.16) is lesser than unity, thus $M_{tot}^{true} < M_{tot}^{dyn}$ as it should be.

However, from real observations, the quantity $\sin\phi$ cannot be obtained for individual double systems, and the dependence of the total mass on r_p and v can be inferred as only statistical. We can estimate the average total mass if we know the mean value of $\sin\phi$. In the 3-dimensional space $\overline{\sin\phi}^{(3)} = 1/2$, and from (2.15)

$$M_{av}^{dyn} \equiv M_{av}^{(3)} = 8\frac{v^2 r_p}{G^{(3)}}. \tag{2.17}$$

The general formula for averaging projections was obtained earlier [11]. In the n-dimensional space,

$$\overline{\sin\phi}^{(n)} = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n+1)/2)}. \tag{2.18}$$

with Γ the Euler gamma-function. Using (2.17) and (2.18), we can obtain for the true average mass

$$M_{av}^{true} \equiv M_{av}^{(n)} = \frac{\pi^{n/2}}{8}\left(\frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\right)^n \left(\frac{R_0}{r_p}\right)^{3-n} M_{av}^{dyn}. \tag{2.19}$$

Clearly, $r_p > R_0$ and the numerical factor in (2.19) is lesser than unity, thus $M_{av}^{true} < M_{av}^{dyn}$.

Note that averaging over angles also affects the criterion of sampling double galaxies. As a criterion, let the galaxies with r_p lesser than some fixed size be considered as double. Since the averaged projection in the space with $2 < n < 3$ is greater than that in the 3-dimensional space, a sample which is made with an assumption $n = 3$ would contain less double galaxies than there are in the real n-dimensional space.

2.3 Dispersions of Velocities in Spheroidal Galaxies

In order to describe the dynamics of a prolongated, in a given case spheroidal, system we should make below some model assumptions. To simplify calculations, the first assumption is that R_0 is small enough in comparison with a characteristic size of a system, so that we can neglect the change of a matter density inside the sphere with the raduis R_0. This is because in any dimensions the potential outside of a homogeneous sphere is equivalent to the potential of a central point with the same mass. Our assumption means that we can consider $\Phi^{(n)}$ instead of Φ^{imp} [see (2.7)]. Preliminary, we consider the isothermal distribution of matter. The base of such a model is the almost constancy of observed velocity dispersions (at least of in inner regions), which serve as a measure of the "temperature" of a system.

Following [12], we can write the expression for isothermal matter density in n dimensions:

$$\rho^{(n)}(r) = \rho_0^{(n)} \exp\left(-\frac{nU^{(n)}(r)}{< v^2 >}\right) = \rho_0^{(n)} \exp\left(-\frac{U^{(n)}(r)}{< v_\parallel^2 >}\right) \tag{2.20}$$

where $U^{(n)}$ is a gravitational energy of the system per umit mass,

$$U^{(n)} = -\Phi^{(n)}, \tag{2.21}$$

$< v_\parallel^2 >$ is the mean square velocity along the line of sight, and the quantity $< v^2 >= n < v_\parallel^2 >$ characterizes the total dispersion of velocities in the n-dimensional system.

It is interesting to note that, first, the parameter n does not enter explicitly the second equality in (2.20) and, second, (2.20) automatically (independently of n) satisfies the virial theorem in n dimensions

$$M^{(n)} < v^2 >= nM^{(n)} < v_\parallel^2 >= \int dM^{(n)}(r)\,\frac{dU^{(n)}}{dr}\,r \tag{2.22}$$

(Stricly speaking, the finiteness of the full mass $M^{(n)}$ is only necessary for the validity of (2.22).)

Consider the n-dimensional version of the Poisson equation [1] in the case of the fully radial dependence of the potential, $\Phi^{(n)} = \Phi^{(n)}(r)$:

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{d}{dr}\Phi^{(n)}\right) = -s^{(n)}G^{(n)}\rho^{(n)} \tag{2.23}$$

Then, using (2.20) and (2.21) in (2.23), we come to the "n-dimensional" Lane–Emden equation (its 3-dimensional version can be found, e.g., in [13])

$$\frac{d^2w}{d^2\xi} + \frac{n-1}{\xi}\frac{dw}{d\xi} = \exp(-w) \tag{2.24}$$

where we have denoted

$$w = \frac{U^{(n)}}{< v_{\|}^2 >}, \qquad \xi = \frac{r}{r_c^{(n)}},$$

and

$$r_c^{(n)} = \sqrt{\frac{< v_{\|}^2 >}{s^{(n)}G^{(n)}\rho^{(n)}}} \tag{2.25}$$

plays the role of a characteristic size.

The equation (2.24) is known to have no explicit solutions. Moreover, the full mass of the isothermal system is infinite. That is why our second assumption is that we take the approximate solution to (2.24) valid for small ξ: $w^{(n)} = (2n)^{-1}\xi^2$, leading to the density distribution

$$\rho^{(n)} = \rho_0^{(n)} \exp\left[-\frac{1}{2n}\left(\frac{r}{r_c^{(n)}}\right)^2\right] \tag{2.26}$$

Since, at any rate some cutoff of the full isothermal distribution were required for real systems, we can use the distribution (2.26) for rough estimates. From now, we consider (2.26) as a model, which indeed has an isothermal inner region.

The full mass of the system is given by an integral

$$M^{(n)} = s^{(n)}\rho_0^{(n)} \int_0^\infty dr\, r^{n-1} \exp\left[-\frac{1}{2n}\left(\frac{r}{r_c^{(n)}}\right)^2\right]$$

$$= \frac{1}{2}(2n)^{n/2}\Gamma\left(\frac{n}{2}\right) s^{(n)}\rho_0^{(n)}(r_c^{(n)})^n = (2n\pi)^{n/2}\rho_0^{(n)}(r_c^{(n)})^n \tag{2.27}$$

For the partial case of 3 dimensions,

$$M^{(3)} = (6\pi)^{3/2}\rho_0^{(3)}(r_c^{(3)})^3 \tag{2.28}$$

Further, we should compare the masses $M^{true} \equiv M^{(n)}$ and $M^{dyn} \equiv M^{(3)}$. To do it, the third assumption is that the characteristic radii of the n-dimensional and 3-dimensional systems are equal:

$$r_c^{(n)} = r_c^{(3)} \equiv r_c, \tag{2.29}$$

Recall that we deal in fact with one system with the same $< v_{\|}^2 >$, but merely make alternative calculations, then (2.29) and (2.25) immediately lead to

$$s^{(n)}\rho_0^{(n)} = 4\pi\rho_0^{(3)}, \tag{2.30}$$

From (2.27) — (2.30), the relation between the true and dynamical masses is

$$M^{true} = \frac{(2n)^{n/2}}{3\sqrt{6\pi}}\,\Gamma\left(\frac{n}{2}\right)\left(\frac{R_0}{r_c}\right)^{3-n} M^{dyn} \tag{2.31}$$

The number factor in (2.31) is easily to be shown lesser than unity for $2 \leq n < 3$; for $n = 2$ its value is ≈ 0.482. Due to that $R_0/r_c \ll 1$, $M^{true} < M^{dyn}$.

Note that, first, the more precise the relation $R_0/r_c \ll 1$ is, the more precisely (2.31) is fulfilled. Second, the special value of the number factor in (2.31) reflects the model chosen, but evidently M^{true} is always smaller than M^{dyn}.

In the three above examples we have considered the change of dimensionality by one leap and obtained the expressions (2.10), (2.19) and (2.31) for the true masses $M^{(n)}$ which explicitly involve the factor $(R_0/r)^{3-n}$.

Now, let us consider the case when the dimensionality changes by several leaps, and suppose that this situation is in accordance with hierarchical distribution of matter in the Universe. Then the chain of relations (2.8) for gravitational constants may correspond to this hierarchy. We see at least five hierarchical levels — outer parts of galaxies, double galaxies, small clusters of galaxies, rich clusters, and superclusters.

Let r_i be some characteristic size of an i-th level ($i = 1, 2, ..., j$). Suppose that $R_{i-1} < r_i < R_i$, then the true masses at each level could be expressed as follows

$$\begin{aligned} M_1^{true} &= M_1^{dyn}\left(\tfrac{R_0}{r_1}\right)^{3-n_1}, \\ M_2^{true} &= M_2^{dyn}\left(\tfrac{R_0}{R_1}\right)^{3-n_1}\left(\tfrac{R_1}{r_2}\right)^{3-n_2}, \\ &\ldots \\ M_j^{true} &= M_j^{dyn}\left(\tfrac{R_0}{R_1}\right)^{3-n_1}\left(\tfrac{R_1}{R_2}\right)^{3-n_2}\ldots\left(\tfrac{R_{j-1}}{r_j}\right)^{3-n_j}. \end{aligned}$$

We see that at each i-th level, there are exactly i factors which can reduce dynamical mass (dark matter) to ordinary luminous matter. Indeed, every fraction R_{i-1}/R_i or R_{i-1}/r_i is lesser than unity. In order these fractions raised to the power $3 - n_i$ remain lesser than unity, it is sufficient to impose $n_i < 3$. Probably, the monotonic fall of dimensionality takes place so that $3 > n_1 > n_2 > \ldots > n_j > n_{j+1} \geq 2$. However, our derivation is too rough, and there are too many undetermined parameters to make any numerical estimates.

3 Relativistic Consideration

Consider the homogeneous and isotropic (n+1)-dimensional Universe with the homogeneous energy density and pressure, $\epsilon(t)$ and $p(t)$, respectively, and the cosmological constant Λ. The behaviour of the Universe is described by the scale factor $a(t)$. The metric interval is as usual

$$ds^2 = c^2dt^2 - a(t)dl^2_{(n)} \tag{3.1}$$

where $dl^2_{(n)}$ is the interval of the n-dimensional space of the constant Gauss curvature; the sign of the later being below denoted by k: $k = 0, \pm 1$. The number n is integer as yet.

The set of the Einstein equations in n dimensions is as follows

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \frac{\kappa^{(n)}}{c^2} \left[\epsilon u_\mu u_\nu - p(g_{\mu\nu} - u_\mu u_\nu)\right] + \Lambda g_{\mu\nu} \tag{3.2}$$

where $\mu, \nu, = 0, 1, \ldots, n$, and $\kappa^{(n)}$ is the Einstein-type constant. It is connected with the Newtonian gravitational constant $G^{(n)}$ [2]:

$$\kappa^{(n)} = \frac{n-1}{n-2} \frac{s^{(n)} G^{(n)}}{c^2} \tag{3.3}$$

where

$$s^{(n)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the surface of the unit-radius n-dimensional ball.

For the interval (3.1), the set (3.2) is converted into the two equations

$$\frac{n(n-1)}{2} \left(\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right) = \kappa^{(n)} \epsilon + \Lambda c^2, \tag{3.4}$$

$$(n-1) \left[\frac{\ddot{a}}{a} + \frac{n-2}{2} \left(\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right)\right] = -\kappa^{(n)} p + \Lambda c^2. \tag{3.5}$$

where the dot denotes differentiation with respect to t.

Equations (3.4) and (3.5) are not independent, and lead to the conservation law

$$\dot{\epsilon} + n \frac{\dot{a}}{a} (\epsilon + p) = 0. \tag{3.6}$$

To solve (3.4) – (3.5) we impose as usual the equation of state

$$p = q\epsilon \tag{3.7}$$

with $q = const$; then the law (3.6) can be written in the form

$$\dot{\epsilon} + n_e \frac{\dot{a}}{a} \epsilon = 0. \tag{3.8}$$

where we have denoted $n_e = n(q + 1)$.

Now, after deriving (3.4)–(3.6), we can make the crucial step: We suppose that n is an arbitrary (noninteger) real number.

Note that the quantity n_e plays the role of an effective dimensionality. From (3.8),

$$\epsilon \propto a^{-n_e}. \tag{3.9}$$

In the FRW framework, it is commonly supposed that q decreases in the process of the expanding Universe, leading to the fact that the effective dimensionality also decreases. In the radiation-dominated epoch, if the dimensionality were $n = 3$, then $q = 1/3$, and $n_e = 4$. [In n dimensions, for radiation $q = 1/n$ and $n_e = n + 1$.] In the present epoch, galaxies can be considered dust-like matter comoving with the expanding Universe because their peculiar velocities are very small $(v/c)^2 = 10^{-6} \ll 1$. Hence we can think $q = 0$ as the real value, and $n_e = n$ from (3.8). However, it is possible that n_e falls off faster owing to not only the fall of q, but also owing to the fall of n.

In considering the static situation ($\dot{a} = 0$) and using (3.7), Eqs. (3.4) and (3.5) become an algebraic set of equations for obtaining Λ and a. For both the signs of k: $k = \pm 1$,

$$\Lambda = \frac{(n_e - 2)}{2} \frac{\kappa^{(n)} \epsilon}{c^2}, \tag{3.10}$$

However, the only choice $k = +1$ (closed Universe) is possible for the positive energy, $\epsilon > 0$. Then,

$$a = \sqrt{\frac{(n-1)}{q+1} \frac{c^2}{\kappa^{(n)} \epsilon}}. \tag{3.11}$$

We see from (3.10) and (3.11) that $\Lambda \neq 0$ is required for staticity when $n_e \neq 2$. In the special case $n_e = 2$, there exists the closed static solution (Einstein-type cosmological model) without cosmological constant ($\Lambda = 0$). However, this leads to the exotic equation of state: $q = -1 + 2/n$. The real equation of state of the modern Universe, $q = 0$, is reached for $n = 2$ only; and we have the simple relation for the scale factor of the 2-dimensional Universe:

$$a = \sqrt{\frac{c^2}{\kappa^{(2)} \epsilon}}. \tag{3.12}$$

If one believes that the dimensionality falls with the growth of relative distance, and hence with the decreasing density in the process of the expanding Universe, then the solution (3.12) might play the role of an asymptotic solution, which will end the expansion of the Universe. It interesting to note that the constant $\kappa^{(2)}$ is immediately connected with the total mass of the closed Universe M_U:

$$\kappa^{(2)} = (4\pi)^{-1} M_U.$$

As in [14], the above consideration can be illustrated from the viewpoint of the Newtonian theory. Let $p = 0$ ($q = 0$). Consider the equation with eliminated curvature which can be obtained as a combination of (3.4) and (3.5):

$$\frac{\ddot{a}}{a} = -\frac{n-2}{n(n-1)} \kappa^{(n)} \epsilon + \frac{2}{n(n-1)} \Lambda c^2. \tag{3.13}$$

Eq.(3.13) has a simple physical sense. If we take (3.3) and write $\epsilon = c^2 \rho$ where ρ is a matter density, then (3.13) can be reduced to the form

$$\ddot{a} = -\frac{G^{(n)} \tilde{M}}{a^{n-1}} + \frac{2}{n(n-1)} \Lambda c^2 \, a \tag{3.14}$$

where we have denoted $\tilde{M} = s^{(n)} \rho a^n / n$. Eq.(3.14) has a Newtonian interpretation, although it is obtained exactly, without any approximations. Let us think that a is the radius of some n-dimensional ball in the n-dimensional space homogeneously filled by matter with the density ρ. Then, $\tilde{M}$ is a constant mass confined inside the ball, whose constancy follows from (3.9). Eq.(3.14) describes the dynamics of a spherical layer of unit mass at the radius a. Evidently, a domain which is external to the ball does not affect the dynamics of the layer. The left-hand side of (3.14) is an acceleration of the layer. The first term on the right-hand side of (3.14) is, in accordance with (1.2), the gravitational attraction force between the layer and the matter in the ball. The second term there is a repulsive force which is linear in a. Thus, the presence of positive Λ provides some cosmic repulsion, which is necessary for staticity in $n \neq 2$ dimensions.

As for the exclusive case $n = 2$, it is easily seen from (3.3) that in order that the constant $\kappa^{(2)}$ be finite, we should suppose that $G^{(n)} \propto (n-2) \cdot Const \to 0$ when $n \to 2$. Thus, the attractive Newtonian force should vanish for $n = 2$, and no compensating repulsive force with $\Lambda \neq 0$ is required for staticity.

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