IMPOSSIBLE SOLUTIONS?

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Abstract

We present vortex-ring current functions and also functions which can be interpreted as the potentials of point sources in n-dimensional (nD) spaces, $n \geq 2$. These functions are the solutions to elliptical equations which involve second-order differential operators generated by applying ∇^2 to (poly)vector quantities. These operators are not obvious Laplacians, we call them anti-Laplacians due to their forms. The presented solutions possesse 3D or 2D asymptotics in spaces with arbitrary (large) odd or even dimensions, respectively. We stress that we deal with the only compact sources, that can suggest some physical speculations. (As to usual nD potentials which are the solutions of Laplace equations, such the behaviour could be provided by some infinitely-stretched sources.) We explicitly construct solutions of homogeneous equations and extract point sources. We have found a transformation from any $(n-2)$D solutions, l.h.s. and r.h.s. of equations to subsequent nD ones, that is a powerfull tool for extracting sources. The latter turn out to be δ-like in the odd-n case, and they are not δ-like but finite in a singular point in the even-n case.

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1 Introduction

It is well-known that in the n-dimensional (nD) space the gravitational and Coulomb-like electric potentials possess the fall-off laws $\sim R^{-(n-2)}$ for $n > 2$, R being a distance from a compact source, i.e., the source located in a confined space domain. In the case $n = 2$, the law has the form $\sim \ln(1/R)$. This basic statement is undoubtedly true. However, we state that in the nD space it is also possible that the fall-off law of a potential-type physical quantity from some compact source is adequate to the law associated with the 3D or 2D space for an arbitrary odd or even n, respectively. – We mean not only a far-zone asymptotics, but a near-zone one as well.

A situation can be explained as follows. Let us imagine that we split the nD space onto 2D and $(n-2)$D subspaces. Consider a function – a geometrical object – with a complicated transformation properties. It does not matter whether it is a scalar, or a vector orthogonal to the radial direction in the 2D space. Let it simultaneously be a component of some polyvector in the $(n-2)$D space. The covariant operator ∇^2 applied to such the function is not obliged to be an obvious scalar Laplace operator, and the solution of the corresponding Poisson-like equation with a compact source is not obliged to be $\propto R^{-(n-2)}$. In the 3D space such a situation is hardly distinguishable. Indeed, the remaining single dimension (denote it by z) in the $(3-2)$D space provides the term $\partial^2/\partial z^2$ in ∇^2, the same for a 1-vector and for a scalar, leading to the solution $\propto R^{-1}$ in both the cases.

The presented study was originated by considering vortex rings in higher (n) dimensions. Generalizations of Stokes equations and solutions for current functions (ψ_n) were obtained. However, as it is known, the construction of a current function is confined by the condition of vanishing divergence of a fluid velocity field. That is why, despite of the mentioned 3D or 2D asymptotics of ψ_n, the asymptotics of the velocity field, the true observable quantity, is n-dependent.

Considering nD vortex rings have lead to considering a class of second-order differential operators, which we call anti-Laplacians, and finding corresponding solutions. Some subclass of this class (anti-double-Laplacians, see below) is just suitable for describing vortex rings. Another subclass (anti-z-Laplacians) can give rise to some physical speculations. Some quantity having the significance, for example, of a potential and satisfying an equation with the anti-z-Laplacian on the l.h.s., and a compact source on the r.h.s., also possesses the 3D or 2D asymptotics. Since a condition of vanishing divergence is not a necessary element of construction, quantities having the significance of forces, velocities, intensities and so on, involving the derivatives of this potential can be determined by such a way as to exhibit 3D or 2D fall-off laws.

A physically meaningful aspect of this situation is that given such the fall-off laws, we cannot determine an exact dimensionality of our physical space. We can only extract a "best-fit" value $n = 2$ or $n = 3$ from observations and only state that the real dimensionality is odd or even. Thus, the effective dimensionality of our space can be dynamical in essence. Perhaps, we are living in an odd-nD space, and such compact sources are just elementary particles, so that we have no contradictions with the physical picture of our world.

Here, the author's intention is not to deal with any special physical models in the light of these speculations. Several mathematical facts which weaken the mentioned basic statement at the beginning are accounted. Further investigations concerning physical and geometric aspects are postponed to future papers.

In Sec.2, we recall the reader how to obtain a current function describing an infinitely thin vortex ring. A traditional derivation of this subject in old and lovely manuals like [1, 2] has a drawback from the modern viewpoint: There is no appellation to δ functions, although the latter is a quite convenient tool. Indeed, the vortex rings had been known well before P.A.M. Dirac invented his δ function. We modify the traditional derivation by including the δ functions. During this derivation we meet, for the first time, an operator belonging to the mentioned class of operators.

In Sec.3, we consider an extension of equations suitable for vortex rings onto nD space. On the l.h.s. of these equations anti-Laplacian operators appear, however, we do not know a priori explicit forms of their r.h.s., i.e., those of ring sources. In this section we only give a constructive way of obtaining some solutions to homogeneous equations. The solutions are finite everywhere except of a ring set of points, and they could serve as those for the nD vortex rings. It turns out that the cases of odd and even n are principally distinguished, and so are their asymptotics. The odd-nD solutions have asymptotics of the 3D ones, whereas the even-nD solutions, omitting some details, have that of 2D ones. It only remains to learn explicit forms of the ring-like sources. However, in order to do it, we have to study some properties of not only anti-Laplacians and but Laplacians as well in the next two sections.

Thus, we return in Sec.4 to the familiar Laplacians and Poisson equations with δ sources possessing some symmetries in relevant coordinate frames, the standard solutions being $\propto R^{-(n-2)}$, certainly. We have found some transformation of an $(n-2)$D solution into a nD one, which is the same for both the odd-nD and even-nD cases. Moreover, the same transformation, from $(n-2)$D to nD, is suitable for both the l.h.s. and r.h.s. of the Poisson equations. Thus, for symmetries considered, we can construct any nD solution if we know the above transformation and the 3D and 2D solutions (or the 4D solution in a special symmetry). In Sec.4, we also show that the traditional-way extraction of an nD δ source from the nD solution gives the same result, which can be obtained via applying our transformation to the 3D or 4D sources.

Sec.5 is an independent study without referring to vortex rings. We examine connections between Laplacians and anti-Laplacians which provide a way of finding solutions similar to those in Sec.2. We pay attention to one of the anti-Laplacians, anti-z-Laplacian, which surprisingly leads to the above mentioned potential-type nD solutions related to a one-point source (with unknown explicit form). It is remarkable that these solutions are somewhat simplified versions of those in Sec.3 and have similar properties. It is also remarkable that, similarly to Sec.4, we also find (another) transformation from $(n-2)$D quantities to nD ones suitable both for solutions and equations. Thus, we construct a transformation machinery which permits one to resolve the problem of finding points sources.

This is done in Sec.6. We obtain two different answers for the cases of odd-n and even-n dimensions. The former case is very simple: Given a 3D source, all the subsequent odd-nD sources are obtainable, and they turn out to be obvious δ-like ones. However, in the latter case the situation is more intricate. In order to input the transformation machinery in the even-n case, we have obliged to have in hand the case $n = 4$. It is simple enough to be calculated immediately, and we construct it in detail. The answer is that the point-like source is not a δ source, but it is finite in a singular point, which contradicts, at first glance, to the known theorem. However, there are no true contradiction; some discussion on this subject is also given.

In Sec.7., we return to vortex-ring-like solutions of Sec.3 and exhibit their sources found with the use of the transformation machinery. We also describe some other ring-like solutions and give several concluding remarks. Three appendices with some necessary mathematical information accomplish our work.

2 Recalling a 3D vortex ring

In the cylindrical coordinate frame (r, φ, z), consider an axially symmetric motion of incompressible fluid. Let the symmetry axis be directed along z, r being the distance from z, and there is no dependence of φ (see [1, 3, 4] or any other suitable manuals). Let $V_r(r, z)$ and $V_z(r, z)$ be the r and z components of a fluid velocity field $\vec{V}$, respectively.

The continuity equation is satisfied everywhere, perhaps except some points:

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial}{\partial r}(rV_r) + \frac{\partial}{\partial z}(rV_z) = 0. \tag{2.1}$$

In our symmetry assumptions, the only φ component of the $\vec{V}$ curl survives, let it be a given value $\Omega_\varphi(r, z)$,

$$(\vec{\nabla} \times \vec{V})_\varphi = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = \Omega_\varphi. \tag{2.2}$$

It follows from (2.1) that the expression

$$rV_z dr - rV_r dz = d\psi$$

is a local differential, where the function $\psi(r, z)$ is called the Stokes current function, and the previous definition of ψ is equivalent to

$$V_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad V_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{2.3}$$

(Our choice of signs in (2.3) coincides with that of [3] and is opposite to that of [1, 4].) Hence, equation (2.2) can be represented as that for ψ:

$$\overline{\Delta}_r \psi \equiv r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -r\Omega_\varphi(r, z). \tag{2.4}$$

Equation (2.4) is usually called the Stokes equation, and the operator defined on its l.h.s. is not a Laplace operator. In keeping in mind further generalizations, we call it anti-r-Laplacian (we refer the reader to Sec.5).

In fact, the function ψ makes the sense of the φ component of a vector field $\vec{A}$. The identically vanishing divergence (2.1) means that $\vec{V}$ is a curl vector, i.e.,

$$\vec{V} = \vec{\nabla} \times \vec{A}.$$

In our assumptions about the vector $\vec{V}$, the only A_φ component of $\vec{A}(r, z)$ is nonzero and

$$V_r = -\frac{1}{r}\frac{\partial A_\varphi}{\partial z}, \qquad V_z = \frac{1}{r}\frac{\partial A_\varphi}{\partial r}. \tag{2.5}$$

The comparison of (2.3) and (2.5) allows us to impose $\psi = A_\varphi$. (This fact has a consequence $\psi = r|\vec{A}|$, which is obviously given in manuals, cf. loc. cit. with our sign convention.) This fact helps one to solve equation (2.4) for ψ by connecting the operator $\overline{\Delta}_r$ with the Laplacian, see below. Our account is given following the familiar way of [2, 1].

In the Cartesian coordinate frame x, y, z, where $x = r\cos\varphi$ and $y = r\sin\varphi$, we take, e.g., the y component of $\vec{A}$:

$$A_y = \frac{\cos\varphi}{r} A_\varphi$$

(we could choose the component A_x that would give the same final result). In the above frame, the vector operator ∇^2 applied to any component of $\vec{A}$ coincides with the scalar operator, denoted by Δ_{Cart}, applied to the same component:

$$\nabla^2 A_y = \Delta_{Cart} A_y \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_y =$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\cos\varphi}{r}\,A_\varphi\equiv\Delta_{(r,\varphi,z)}\frac{\cos\varphi}{r}\,A_\varphi.$$

Then, we replace back A_φ by ψ, and it is easy to establish the validity of the rearrangement

$$\Delta_{(r,\varphi,z)}\frac{\cos\varphi}{r}\,\psi=\frac{\cos\varphi}{r}\,\overline{\Delta}_r\psi. \tag{2.6}$$

We stress once more that equation (2.6) is an important step which allows one to construct a solution for ψ (or A_φ) generated by a δ-like source using the fundamental solution of the Poisson equation.

Consider an infinitely thin vortex ring of the radius a but with the finite vortex intensity κ: $\kappa=\int\Omega_\varphi dS$ where $dS=dr\,dz$ is the square of the infinitesimal ring cross section. We should impose

$$\Omega_\varphi=\kappa a\,\frac{\delta(r-a)}{r}\,\delta(z),$$

leading to equation (2.4) in the form

$$\overline{\Delta}_r\psi=-\kappa a\,\delta(r-a)\delta(z). \tag{2.7}$$

From (2.6) and (2.7), we obtain the Poisson-like equation:

$$\Delta_{(r,\varphi,z)}\,\frac{\cos\varphi}{r}\,\psi=-\kappa a\,\frac{\cos\varphi}{r}\,\delta(r-a)\delta(z). \tag{2.8}$$

Due to the compact character of the source on the r.h.s. of (2.8) the solution can be found by a standard way:

$$\frac{\cos\varphi}{r}\,\psi=\frac{\kappa a}{4\pi}\int\limits_0^\infty dr'\,r'\int\limits_0^{2\pi}d\varphi'\int\limits_{-\infty}^{\infty}dz'\,\frac{\cos\varphi'}{r'}\,\frac{\delta(r'-a)\,\delta(z')}{|\vec{R}-\vec{R}'|} \tag{2.9}$$

where

$$|\vec{R}-\vec{R}'|=[r^2+r'^2-2r'r\cos(\varphi-\varphi')+(z-z')^2]^{1/2}$$

is the distance between the points with the coordinates (r,φ,z) and (r',φ',z'). Integrating (2.9) with respect to r' and z' gives

$$\frac{\cos\varphi}{r}\,\psi=\frac{\kappa a}{4\pi}\int\limits_0^{2\pi}d\varphi'\frac{\cos\varphi'}{[r^2+a^2-2ar\cos(\varphi-\varphi')+z^2]^{1/2}}\,. \tag{2.10}$$

In order to remove the factor $\cos\varphi$ on the l.h.s. of (2.10), we displace the origin of φ': $\varphi'=\varphi+\alpha$, so that integration will be done with respect to α. After using the equality

$$\cos\varphi'=\cos\varphi\,\cos\alpha-\sin\varphi\,\sin\alpha$$

and ensuring that

$$\int\limits_0^{2\pi}d\alpha\,\frac{\sin\alpha}{[r^2+a^2-2ar\cos\alpha+z^2]^{1/2}}=0,$$

the above factor on the l.h.s. of (2.10) cancels with that on a r.h.s. of a final expression.

The required solution for the 3D vortex ring can now be written in the form

$$\psi=\frac{\kappa a}{2\pi}\,r\int\limits_0^{\pi}d\alpha\,\frac{\cos\alpha}{[r^2+a^2-2ar\cos\alpha+z^2]^{1/2}}\,. \tag{2.11}$$

And here we stop as yet. Our task was to only obtain (2.11) for comparison with further extensions on multidimensional spaces. It is well known how to express ψ via elliptical functions, there are also known its various properties and asymptotics, see, e.g., already cited manuals and many others. Note that instead of the vortex ring, we could speak about a ring electric contour, $A_\varphi(=\psi)$ being the φ component of the electromagnetic vector potential. — There are no difference from the mathematical viewpoint.

3 The nD vortex-ring-like solutions

In order to describe an n-dimensional ring structure, which is a direct product of the ring of the radius a on an infinitesimal $(n-1)$-dimensional ball, it is convenient to introduce an (r, z) coordinate frame corresponding to the $2+(n-2)$ splitting of R^n: $R^n = R^2 \times R^{n-2}$. In each space, R^2 and R^{n-2}, we construct spherical coordinate frames with the radial coordinates r and z, and the angle coordinates φ and $\theta_1, \theta_2, \ldots, \theta_{n-3}$, respectively (see Appendix A).

From now we shall not confine ourselves by any physical interpretation, and we omit any physically meaningful coefficients. Consider a vector field $\vec{v}(r, z)$ which has the only $v_r(r, z)$ and $v_z(r, z)$ nonvanishing components. Let the divergence of this vector field be equal to zero almost everywhere,

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{z^{n-3}}\frac{\partial}{\partial z}(z^{n-3} v_z) = 0 \tag{3.1}$$

[see (A.5), also note that in the frame selected $v^z = v_z$, $v^r = v_r$]. If we impose, as before in Sec.2,

$$v_r = -\frac{1}{r z^{n-3}}\frac{\partial \psi}{\partial z}, \qquad v_z = \frac{1}{r z^{n-3}}\frac{\partial \psi}{\partial r}, \tag{3.2}$$

then the divergence (3.1) vanishes identically.

The extension of the curl operator to the nD space leads to a covariant 2-vector or a contravariant $(n-2)$-vector, see, e.g., [5]. However, this fact does not prevent us to propose that its only nonvanishing covariant component is unknown as yet value $\omega_{r,z}$:

$$\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = \omega_{r,z}. \tag{3.3}$$

After substituting (3.2) into (3.3), an operator can be defined which we call anti-double-Laplacian and denote by $\overline{\overline{\Delta}}^{(n)}$:

$$\overline{\overline{\Delta}}^{(n)}\psi_n = \left(r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + z^{n-3}\frac{\partial}{\partial z}\frac{1}{z^{n-3}}\frac{\partial}{\partial z}\right)\psi_n = -r z^{n-3}\omega_{r,z}. \tag{3.4}$$

Now we shall find solutions to the homogeneous (anti-double-Laplace) equations for all $n \geq 3$,

$$\overline{\overline{\Delta}}^{(n)}\psi_n = 0, \tag{3.5}$$

keeping in mind that $\omega_{r,z}$ should be considered equal to zero almost everywhere, except of the set of points $r = a$, $z = 0$, because we suggest to search for the vortex-ring-like solutions. This is done below, separately for odd and even n.

3.1 The odd-nD solutions

Consider the probe function

$$\overline{\psi}_k = r \int_0^\pi d\alpha \, \cos\alpha \, \frac{z^{k-3}}{R^{k-2}} \tag{3.6}$$

where we denote

$$R = \sqrt{\rho^2 + z^2}, \tag{3.7a}$$

$$\rho^2 = r^2 + a^2 - 2ar\cos\alpha. \tag{3.7b}$$

In applying the operator $\overline{\overline{\Delta}}^{(n)}$ to $\overline{\psi}_k$, we come to the equality

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_k = r \int_0^\pi d\alpha \, \cos\alpha \left[-(k-3)(n+1-k)\frac{z^{k-5}}{R^{k-2}} + (n-k)(k-2)\,\frac{z^{k-3}}{R^k} \right] \tag{3.8}$$

after using the integral identity

$$\int_0^\pi d\alpha \, \cos\alpha \left[\frac{1}{R^{k-2}} - (k-2)\,\frac{ar\cos\alpha}{R^k} + k(k-2)\,\frac{a^2r^2\sin^2\alpha}{R^{k+2}} \right] =$$

$$\int_0^\pi d\alpha \, \frac{\partial}{\partial\alpha} \left[\frac{\sin\alpha}{R^{k-2}} - (k-2)\,\frac{ar\sin\alpha\cos\alpha}{R^k} \right] = 0. \tag{3.9}$$

Let us explicitly write the formulae (3.8) for odd integer k between 3 and n:

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_3 = r \int_0^\pi d\alpha \cos\alpha \, (n-3)\,\frac{1}{R^3}, \tag{3.10a}$$

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_5 = r \int_0^\pi d\alpha \cos\alpha \left[-2(n-4)\,\frac{1}{R^3} + 3(n-5)\frac{z^2}{R^5} \right], \tag{3.10b}$$

$$\cdots$$

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_{n-2} = r \int_0^\pi d\alpha \cos\alpha \left[-3(n-5)\,\frac{z^{n-7}}{R^{n-4}} + 2(n-4)\,\frac{z^{n-5}}{R^{n-2}} \right], \tag{3.10c}$$

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_n = r \int_0^\pi d\alpha \cos\alpha \left[-(n-3)\,\frac{z^{n-5}}{R^{n-2}} \right]. \tag{3.10d}$$

We see that the terms with the factors $(k-3)$ and $(n-k)$ have disappeared on the r.h.s. of (3.10a) and (3.10d) for $k=3$ and $k=n$, respectively. The remaining terms can be compensated when taking suitable combinations of $\overline{\psi}_k$: The single term ($\propto 1/R^3$) in (3.10a) can be compensated by the first term in (3.10b), and so on, the last ($\propto z^{n-5}/R^{n-2}$) term in (3.10c) can be finally compensated by the single term in (3.10d). Thus, the desired solution can be constructed as a finite series of $\overline{\psi}_k$,

$$\psi_n = {\sum_{k=3}^{n}}' \, a_{k,n}\overline{\psi}_k = r \int_0^\pi d\alpha \cos\alpha \, {\sum_{k=3}^{n}}' \, a_{k,n}\,\frac{z^{k-3}}{R^{k-2}}\,. \tag{3.11}$$

where the prime over the sum indicates that the latter is taken with respect to the only odd k. As to the coefficients $a_{k,n}$, the recurrence relation takes place:

$$a_{k+2,n} = a_{k,n}\,\frac{(k-2)(n-k)}{(k-1)(n-k+1)}\,. \tag{3.12}$$

Solving (3.12) with the (arbitrary as yet) choice $a_{3,n} = 1$ for every n and $k \geq 5$ yields

$$a_{k,n} = \frac{1\cdot 3\cdot 5\cdots(k-4)(n-3)(n-5)\cdots(n-k+2)}{2\cdot 4\cdot 6\cdots(k-3)(n-4)(n-6)\cdots(n-k+1)}\,. \tag{3.13}$$

More information about the properties of the coefficients $a_{k,n}$ can be found in Appendix B.

In addition to already given solution (2.11) for $n = 3$, we expose here the three subsequent solutions (3.11) for $n = 5, 7, 9$:

$$\begin{aligned}
\psi_3 &= r\int_0^\pi d\alpha\cos\alpha\,\frac{1}{R}\,,\\
\psi_5 &= r\int_0^\pi d\alpha\cos\alpha\left(\frac{1}{R}+\frac{z^2}{R^3}\right),\\
\psi_7 &= r\int_0^\pi d\alpha\cos\alpha\left(\frac{1}{R}+\frac{2}{3}\,\frac{z^2}{R^3}+\frac{z^4}{R^5}\right),\\
\psi_9 &= r\int_0^\pi d\alpha\cos\alpha\left(\frac{1}{R}+\frac{3}{5}\,\frac{z^2}{R^3}+\frac{3}{5}\,\frac{z^4}{R^5}+\frac{z^6}{R^7}\right).
\end{aligned}$$

The far-zone asymptotics of the obtained solutions is very interesting from the physical viewpoint. When $r \to \infty$ for any finite z, the integrand of the term $\overline{\psi}_k$ falls off as $\sim r^{2-k/2}$, so that for every odd-nD solution independently of n the term with $k = 3$, $1/R$, is asymptotically a leading one. When $z \to \infty$ for any finite r, all the $\overline{\psi}_k$ integrands have the same asymptotics $\sim z^{-1}$. Thus, far from a compact source, the nD solution ψ_n behaves almost like the 3D solution.

It is also an interesting common feature of all these solutions that in the 2D plane $z = 0$ they exactly exhibit the true 3D behaviour. This fact determines the asymptotics $z \to 0$. While for every finite z the integrand of any ψ_n is always $\sim z^{-1}$ when $\rho \to 0$. — This is also the near 3D behaviour, leading to the 3D near-zone asymptotics.

3.2 The even-nD solutions

It is clear from the consideration in the previous subsection that any function of the (3.6)-kind is not suitable for the even-D case, hence we probe other functions. For even integers $l > 4$, we define

$$\overline{\psi}_l = r\int_0^\pi d\alpha\cos\alpha\,\frac{z^{l-4}}{R^{l-4}}\,, \tag{3.14}$$

whereas for $l = 4$, we give a distinct definition

$$\overline{\psi}_4 = r\int_0^\pi d\alpha\cos\alpha\,\ln\frac{1}{R}\,. \tag{3.15}$$

We can find that, for $l > 4$

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_l = r\int_0^\pi d\alpha \cos\alpha\,(l-4)\left[-(n+2-l)\,\frac{z^{l-6}}{R^{l-4}} + (n-l)\,\frac{z^{l-4}}{R^{l-2}}\right], \tag{3.16}$$

after using the integral identity (3.9) with k replaced by l, and for $l = 4$

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_4 = r\int_0^\pi d\alpha \cos\alpha\left[+(n-4)\,\frac{1}{R^2}\right] \tag{3.17}$$

after using the following integral identity

$$\int_0^\pi d\alpha\,\frac{\cos\alpha}{R^4}\left(\ln\frac{1}{R} - \frac{ar\cos\alpha}{R^2} + 2\,\frac{a^2r^2\sin^2\alpha}{R^4}\right) =$$

$$\int_0^\pi d\alpha\,\frac{\partial}{\partial\alpha}\left(\sin\alpha\ln\frac{1}{R} - \frac{ar\sin\alpha\cos\alpha}{R^2}\right) = 0. \tag{3.18}$$

First of all, note that $\psi_4 = \overline{\psi}_4$ is a solution to (3.5) for $n = 4$, which is clear from (3.17): $\overline{\overline{\Delta}}^{(4)}\psi_4 = 0$. As before, we also explicitly write (3.16):

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_6 = r\int_0^\pi d\alpha \cos\alpha\, 2\left[-(n-4)\,\frac{1}{R^2} + (n-6)\,\frac{z^2}{R^4}\right], \tag{3.19a}$$

$$\cdots$$

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_{n-2} = r\int_0^\pi d\alpha \cos\alpha\,(n-6)\left[-4\,\frac{z^{n-8}}{R^{n-6}} + 2\,\frac{z^{n-6}}{R^{n-4}}\right], \tag{3.19b}$$

$$\overline{\overline{\Delta}}^{(n)}\overline{\psi}_n = r\int_0^\pi d\alpha \cos\alpha\,(n-4)\left[-2\,\frac{z^{n-6}}{R^{n-4}}\right]. \tag{3.19c}$$

For $n \neq 4$, the appearance of (3.17) and (3.19) also demonstrates that some linear combination of $\overline{\psi}_l$ can give the solution to (3.5): The term $\sim 1/R^2$ in (3.17) can be compensated by the first term in (3.19a), and so on, the last term in (3.19b) being compensated by the single term in (3.19c). The solution can be written in the form

$$\psi_n = a_{4,n}\overline{\psi}_4 + \sum_{l=6}^{n}{}'' \; a_{l,n}\overline{\psi}_l \tag{3.20}$$

where the double prime means that the sum is taken with respect to the only even l, and there is a simple recurrence relation for $a_{l,n}$ when $l \geq 6$:

$$a_{l+2,n} = a_{l,n}\,\frac{l-4}{l-2}\,. \tag{3.21}$$

If we choose $a_{4,n} = 1$, then $a_{6,n} = 1/2$ from (3.17) and (3.19a), and (3.21) can be easily resolved for $l \geq 6$:

$$a_{l,n} = \frac{1}{l-4}\,. \tag{3.22}$$

As a result of (3.20) and (3.22), the function

$$\psi_n = r\int_0^\pi d\alpha\cos\alpha\left(\ln\frac{1}{R}+\sum_{l=6}^{n}{}''\,\frac{1}{l-4}\,\frac{z^{l-4}}{R^{l-4}}\right) \tag{3.23}$$

for $n>4$ is the solution we just searched for.

Here it is pertinent to consider the most degenerate case $n=2$ where the "z part" of the operator (3.4) is absent, that is

$$\overline{\overline{\Delta}}^{(2)}\psi_2 = \overline{\Delta}_r^{(2)}\psi_2 = r\,\frac{\partial}{\partial r}\,\frac{1}{r}\,\frac{\partial}{\partial r}\,\psi_2 = 0. \tag{3.24}$$

As expected, the function

$$\psi_2 = r\int_0^\pi d\alpha\cos\alpha\;\ln\frac{1}{\rho}\,. \tag{3.25}$$

is a vortex-ring-like solution to (3.24) with ρ defined by (3.7b); during this derivation the integral identity similar to (3.18) is used:

$$\int_0^\pi d\alpha\cos\alpha\left(\ln\frac{1}{\rho}-\frac{ar\cos\alpha}{\rho^2}+2\,\frac{a^2r^2\sin^2\alpha}{\rho^4}\right)=$$

$$\int_0^\pi d\alpha\,\frac{\partial}{\partial\alpha}\left(\sin\alpha\ln\frac{1}{\rho}-\frac{ar\sin\alpha\cos\alpha}{\rho^2}\right)=0.$$

From the physical viewpoint, the solution (3.25) can be hardly interpreted as a proper vortex ring because there is no an orthogonal direction (z) to provide rotation around the ring axis. Nevertheless, it is worth including in our collection because it is required for a study described in Sec.7.

The distinct feature of the solution (3.23) is that the coefficients $a_{l,n}$ do not involve n, so that the $(n+2)$D solution merely acquires an extra term as compared with the nD one. For example, the 4D solution and the three subsequent solutions with $l\geq 6$ are as follows

$$\psi_4 = r\int_0^\pi d\alpha\cos\alpha\ln\frac{1}{R}\,,$$

$$\psi_6 = r\int_0^\pi d\alpha\cos\alpha\left(\ln\frac{1}{R}+\frac{1}{2}\,\frac{z^2}{R^2}\right),$$

$$\psi_8 = r\int_0^\pi d\alpha\cos\alpha\left(\ln\frac{1}{R}+\frac{1}{2}\,\frac{z^2}{R^2}+\frac{1}{4}\,\frac{z^4}{R^4}\right),$$

$$\psi_{10} = r\int_0^\pi d\alpha\cos\alpha\left(\ln\frac{1}{R}+\frac{1}{2}\,\frac{z^2}{R^2}+\frac{1}{4}\,\frac{z^4}{R^4}+\frac{1}{6}\,\frac{z^6}{R^6}\right).$$

From the appearance of the solutions (3.15), (3.23) [and (3.25)], a far asymptotics is easily seen because the term $\ln(1/R)$ is always leading: When $r\to\infty$, the integrands of all the ψ_n are $\sim\ln(1/r)$ starting from $n=2$ independently of n, and when $z\to\infty$, they are $\sim\ln(1/z)$ starting from $n=4$ independently of n as well. This is just the 2D asymptotics.

In the hypersurface $z = 0$, all these nD solutions exactly behave like the 2D ones, $\sim \ln(1/\rho)$; whereas for every finite z, the integrand of ψ_n is always $\sim \ln(1/z) + Const$ when $\rho \to 0$. Thus we can conclude about the almost 2D near-zone behaviour as well.

However, it should be noted that due to (3.2), the asymptotical behaviour of v_r and v_z for both odd and even n carries an n depenence.

4 Known and unknown properties of Laplacians and relevant quantities

4.1 The case of spherical symmetry

It is widely known that the scalar Poisson equation in n dimensions with the unit point source

$$\Delta^{(n)}\Phi_n = -\sigma_n \delta(\vec{R}) \equiv s(\vec{R}) \tag{4.1}$$

has the solutions

$$\Phi_n = \frac{1}{n-2}\,\frac{1}{|\vec{R}|^{n-2}}\,, \quad \text{for} \quad n \geq 2,$$

$$\Phi_2 = \ln\frac{1}{|\vec{R}|}\,, \quad \text{for} \quad n = 2,$$

where $\vec{R}$ is the radius-vector originating from the point source, and

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{4.2}$$

with Γ the gamma function is the surface of the $(n-1)$-dimensional sphere of unit radius. The expressions for equation (4.1) and its solutions are indeed covariant, without referring to any coordinate frame.

First of all we shall consider the spherical coordinate frame (the x frame), see Appendix A. This is the simplest case, but it is very illustrative and useful for further studies. In the x frame, the entire expression for the Laplacian involving all the $n-1$ angles is given by (A.2). Consider the spherically symmetric case so that $\Phi_n(\vec{x}) = \Phi_n(x)$ and the only radial dependence in (A.2) survives. We also replace $\delta(\vec{x})$ on $\delta(x)$ using the formula

$$\delta(\vec{x}) = \frac{\delta_+(x)}{\sigma_n x^{n-1}} \tag{4.3}$$

provided by the definition

$$\int_{\tau^{(n)}} d\tau^{(n)} \delta(\vec{x})\phi(x) = \int\limits_0^\infty dx\ \delta_+(x)\phi(x) = \phi(0). \tag{4.4}$$

(For accuracy, we add the index "+" to δ, because one often defines $\int_0^\infty dx\ \delta(x)\phi(x) = \phi(0)/2$ for the singular point $x = 0$ coinciding with a limit of integration, see, e.g., [6].)

The truncated equation (4.1) now acquires the form

$$\Delta_x^{(n)}\Phi_n(x) \equiv \frac{1}{x^{n-1}}\,\frac{\partial}{\partial x}\,x^{n-1}\,\frac{\partial}{\partial x}\,\Phi_n(x) = -\,\frac{\delta_+(x)}{x^{n-1}} \equiv s_x^{(n)}(x), \tag{4.5}$$

and has the solutions

$$\Phi_n(x) = \frac{1}{n-2}\,\frac{1}{x^{n-2}}\,, \quad \text{for} \quad n \geq 2, \tag{4.6a}$$

$$\Phi_2(x) = \ln\frac{1}{x}\,, \quad \text{for} \quad n = 2. \tag{4.6b}$$

Now consider a transformation which converts the $(n-2)$D solution (4.6), $\Phi_{n-2}(x)$, into the nD one, $\Phi_n(x)$. For $n \geq 4$:

$$\Phi_n(x) = -\frac{1}{n-2}\,\frac{1}{x}\,\frac{\partial}{\partial x}\,\Phi_{n-2}(x). \tag{4.7}$$

The equality (4.7) is evident. We see that odd-nD solutions are transformed into odd-nD ones starting from $n = 5$, and even-nD solutions are transformed into even-nD ones starting from $n = 4$. In other words, every nD solution can be obtained in applying the above transformation by required number of times to Φ_3 or to Φ_2.

Moreover, taking into account the following operator rearrangement

$$\frac{1}{x}\,\frac{\partial}{\partial x}\,\Delta_x^{(n-2)} = \Delta_x^{(n)}\,\frac{1}{x}\,\frac{\partial}{\partial x} \tag{4.8}$$

we obtain

$$\Delta_x^{(n)}\Phi_n(x) = -\frac{1}{n-2}\,\frac{1}{x}\,\frac{\partial}{\partial x}\,\Delta_x^{(n-2)}\Phi_{n-2}(x). \tag{4.9}$$

The equality (4.9) means that the l.h.s. of the Poisson equation (4.5) is transformed similar to the solution itself.

Independently of (4.9), the direct calculation with the explicit form of $s^{(n-2)}(x)$ [see the definition on the r.h.s. of (4.5)] shows that the similar situation takes place for sources:

$$s_x^{(n)}(x) = -\frac{1}{n-2}\,\frac{1}{x}\,\frac{\partial}{\partial x}\,s_x^{(n-2)}(x) \tag{4.10}$$

after using the formal equality $x\delta'_+(x) = -\delta_+(x)$ where the prime denotes differentiation.

4.2 The case of double-spherical symmetry

Now let us return to the (r, z) coordinate frame introduced before in Sec.3, see also Appendix A. In the (r, z) frame, the central point source in (4.1) for the double-spherical symmetry can be expressed as follows $(n > 2)$

$$s^{(n)}(\vec{R}) = -\sigma_n\delta(\vec{r})\delta(\vec{z}) = -\sigma_n\,\frac{\delta_+(r)}{\sigma_2 r}\,\frac{\delta_+(z)}{\sigma_{n-2}z^{n-3}} = -\frac{1}{n-2}\,\frac{\delta_+(r)}{r}\,\frac{\delta_+(z)}{z^{n-3}}\,, \tag{4.11}$$

where we have used (4.3), and the equality

$$\sigma_n = \frac{\sigma_2\,\sigma_{n-2}}{n-2} \tag{4.12}$$

which is easy to verify with referring to the definition (4.2), also note that $\sigma_2 = 2\pi$ and $\sigma_1 = 2$.

For the (r, z) dependence of a solution, the truncated operator (A.6) combined with the source (4.11) gives the Poisson equation $(n > 2)$

$$\Delta_{r,z}^{(n)}\Xi_n(r,z) \equiv \left(\frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} + \frac{1}{z^{n-3}}\,\frac{\partial}{\partial z}\,z^{n-3}\,\frac{\partial}{\partial z}\right)\Xi_n(r,z) = -\frac{\delta_+(r)}{r}\,\frac{\delta_+(z)}{z^{n-3}} \equiv s_{r,z}^{(n)} \tag{4.13a}$$

with the solution

$$\Xi_n(r,z) = \frac{1}{\bar{R}^{n-2}}\,. \tag{4.14a}$$

where

$$\bar{R} = \sqrt{r^2+z^2}.$$

Note that due to the remarkable formula (4.12), the factor $(n-2)$ is removed both on the r.h.s. of (4.13a) and in the solution (4.14a).

The case $n = 2$ is degenerate because there are no the z space, and the Poisson equation

$$\Delta^{(n)}_{r,z}\Xi_2(r) = \Delta^{(2)}\Xi_2(r) = \frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}\,\Xi_2 = -\,\frac{\delta_+(r)}{r} \tag{4.13b}$$

has the trivial solution

$$\Xi_2 = \ln\frac{1}{r}\,. \tag{4.14b}$$

In the case of the (r,z) frame, we can also introduce a transformation with properties similar to the above one despite of the presence of the two variables, r and z. It is easy to verify that the required transformation is as follows ($n \geq 5$)

$$\Xi_n(r,z) = -\,\frac{1}{n-4}\,\frac{1}{z}\,\frac{\partial}{\partial z}\,\Xi_{n-2}(r,z) \equiv f_L^{(n,n-2)}\Xi_{n-2}(r,z). \tag{4.15}$$

The appearance of the factor $(n-4)$ instead of $(n-2)$ in (4.15) is due to that the solutions (4.6a) and (4.14a) have distinct coefficients. As before, the odd solutions are obtainable from the odd ones starting from $n = 5$. However unlike the previous case, our transformation works starting from $n = 6$ for even solutions. The point is that, as mentioned, the z dependence of the Laplace operator and the corresponding solution disappears for $n = 2$, see (4.14b), that is why Ξ_4 cannot be obtained from Ξ_2.

Here, the rearrangement of the (4.8)-type also exists,

$$\frac{1}{z}\,\frac{\partial}{\partial z}\,\Delta^{(n-2)}_{r,z} = \Delta^{(n)}_{r,z}\,\frac{1}{z}\,\frac{\partial}{\partial z}\,,$$

and leads to the equality analogous to (4.9):

$$\Delta^{(n)}_{r,z}\Xi_n = f_L^{(n,n-2)}\Delta^{(n-2)}_{r,z}\Xi_{n-2}. \tag{4.16}$$

For the δ-source in (4.13a), the independent of (4.16) direct calculation also gives

$$s^{(n)}_{r,z} = f_L^{(n,n-2)}s^{(n-2)}_{r,z}. \tag{4.17}$$

Let us call the set of quantities Ξ_n, $\Delta^{(n)}_{r,z}\Xi_n$, and $s^{(n)}_{r,z}$ the Laplace set, L_n:

$$L_n = \{\Xi_n,\ \Delta^{(n)}_{r,z}\Xi_n,\ s^{(n)}_{r,z}\}, \tag{4.18}$$

so that the equalities (4.15), (4.16) and (4.17) acquire the compact notation

$$L_n = f_L^{(n,n-2)}L_{n-2}. \tag{4.19}$$

Equation (4.9) can be resolved, in the sense that any nD quantity from the Laplace set can be expressed as below. For odd and even n,

$$L_n = f_L^{(n,n-2)}f_L^{(n-2,n-4)}\ldots f_L^{(5,3)}L_3 = \frac{1}{(n-4)!!}\left(-\,\frac{1}{z}\,\frac{\partial}{\partial z}\right)^{(n-3)/2}L_3 \tag{4.20}$$

and

$$L_n = f_L^{(n,n-2)}f_L^{(n-2,n-4)}\ldots f_L^{(6,4)}L_4 = \frac{1}{(n-4)!!}\left(-\,\frac{1}{z}\,\frac{\partial}{\partial z}\right)^{(n-4)/2}L_4, \tag{4.21}$$

respectively.

4.3 Methodological extraction of δ-sources

In methodological goals and for future applications in Sec.6, we show how to prove that the nD solutions (4.14a) really provide corresponding δ sources, i.e., to prove (4.13a). We shall demonstrate it by two ways. The first of them can be found in any suitable manual in functional analysis, see e.g., [7]. This is very trivial derivation in the case of the x frame, however it has some subtleties in the case of the (r, z) frame. The second way, just invented, consists in using the introduced transformation $f_L^{(n,n-2)}$ in order to construct a proof by a recurrent procedure.

Consider a function $\phi(r, z) \ni \mathcal{D}(\mathcal{G})$, where $\mathcal{D}(\mathcal{G})$ denotes the class of trial functions, i.e. that of finite $C^\infty(\mathcal{G})$ functions in the domain $\mathcal{G}(r, z)$: $supp\, \phi \subset \mathcal{G}(r, z)$. Then,

$$\int_{\tau^{(n)}} d\tau^{(n)} \phi\, \Delta_{r,z}^{(n)} \Xi_n = \int_{\tau^{(n)}} d\tau^{(n)} (\Delta_{r,z}^{(n)} \phi)\, \Xi_n \tag{4.22}$$

where $d\tau^{(n)} = d\sigma_2 d\sigma_{n-2} r z^{n-3} dr\, dz$, see (A.4), and integration is performed over a large enough volume $\tau^{(n)}$, so that the ranges of values of the r and z coordinates contain $\mathcal{G}(r, z)$.

Traditional way. Following [7] and other manuals, we can write

$$\begin{aligned}\int_{\tau^{(n)}} d\tau^{(n)} (\Delta_{r,z}^{(n)} \phi)\, \Xi_n &= \lim_{\varepsilon,\eta\to 0} \int_{\tau^{(n)}_{\varepsilon,\eta}} d\tau^{(n)} (\Delta_{r,z}^{(n)} \phi)\, \Xi_n \\ &= \lim_{\varepsilon,\eta\to 0} \left(\int_{\tau^{(n)}_{\varepsilon,\eta}} d\tau^{(n)} \phi\, \Delta_{r,z}^{(n)} \Xi_n + J_r + J_z \right)\end{aligned} \tag{4.23}$$

using the property (4.22), where the volume $\tau^{(n)}_{\varepsilon,\eta}$ means that we exclude from $\tau^{(n)}$ the small domain near the point $r = 0$, $z = 0$; $\tau^{(n)}_{\varepsilon,\eta}$: $r \geq \varepsilon$, $z \geq \eta$ (or $|z| \geq \eta$ in the 3D case). The quantities J_r and J_z denote the surface integrals:

$$J_r = J_{r1} + J_{r2},$$

$$J_{r1} = \int_{S_r} dS_r \left[\phi\, \frac{\partial \Xi_n}{\partial z} \right]_{z=\eta} = -(n-2)\sigma_2\sigma_{n-2}\eta^{n-2} \int_0^\varepsilon dr\, r\phi(r,\eta)\, \frac{1}{(r^2+\eta^2)^{n/2}}\,, \tag{4.24}$$

$$J_{r2} = -\int_{S_r} dS_r \left[\frac{\partial \phi}{\partial z}\, \Xi_n \right]_{z=\eta} = -\sigma_2\sigma_{n-2}\eta^{n-2} \int_0^\varepsilon dr\, r\, \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} \frac{1}{(r^2+\eta^2)^{(n-2)/2}}\,, \tag{4.25}$$

and

$$J_z = J_{z1} + J_{z2},$$

$$J_{z1} = \int_{S_z} dS_z \left[\phi\, \frac{\partial \Xi_n}{\partial r} \right]_{r=\varepsilon} = -(n-2)\sigma_2\sigma_{n-2}\varepsilon^2 \int_0^\eta dz\, z^{n-3}\phi(\varepsilon, z)\, \frac{1}{(\varepsilon^2+z^2)^{n/2}}\,, \tag{4.26}$$

$$J_{z2} = -\int_{S_z} dS_z \left[\frac{\partial \phi}{\partial r}\, \Xi_n \right]_{r=\varepsilon} = -\sigma_2\sigma_{n-2}\varepsilon^2 \int_0^\eta dz\, z^{n-3}\, \left.\frac{\partial \phi}{\partial r}\right|_{r=\varepsilon} \frac{1}{(\varepsilon^2+z^2)^{(n-2)/2}}\,, \tag{4.27}$$

where

$$dS_r = d\sigma_2 d\sigma_{n-2} dr\, r\eta^{n-3}$$

and

$$dS_z = d\sigma_2 d\sigma_{n-2} dz\, \varepsilon z^{n-3}$$

are the $(n-1)$-dimensional surface elements. The 3-dimensional analogue of S_r is the base surface of the cylinder $0 \le r \le \varepsilon$, $-\eta \le z \le \eta$, and that of S_z is the lateral surface of the same cylinder. (Note that a 0-dimensional sphere here consists of the two isolated points $z = -\eta$ and $z = \eta$.)

In returning to (4.23), we note, first of all, that the prelimit term involving $\Delta^{(n)}_{r,z}\Xi_n$ vanishes. Further, there are two ways of making a limit procedure depending on what limit (with respect to ε or η) is taken first. From the expressions (4.24), (4.25) and (4.26), (4.27) we see that always

$$\lim_{\varepsilon\to 0} J_r = 0, \quad \lim_{\eta\to 0} J_z = 0.$$

This fact means that when the order of taking limits in (4.23) is $\varepsilon \to 0$ and then $\eta \to 0$ (or $\eta \to 0$, and then $\varepsilon \to 0$) the only integral J_z (or J_r) "works".

It can be easily shown that the following limit expressions of the integrals containing the derivatives of ϕ vanish

$$\lim_{\eta\to 0} J_{r2} = 0, \quad \lim_{\varepsilon\to 0} J_{z2} = 0.$$

The calculation of the remaining integrals (4.24) and (4.26) will be exhibited in more details. The trivial integration in (4.24), mutual for both odd-nD and even-nD cases, leads to

$$\lim_{\eta\to 0} J_{r1} = -\sigma_2\sigma_{n-2}\bar{\phi}_\varepsilon$$

with $\bar{\phi}_\varepsilon$ some average value of $\phi(r,0)$ over the interval $0 \le r \le \varepsilon$.

As to the integral in (4.26), we should use different handbook integrals for the odd-nD and even-nD cases. For odd n, the integral in (4.26) transforms to (C.1). For even n the above integral acquires the form (C.2) and in the $\varepsilon \to 0$ limit we additionally require the expression (C.3). However, in both the odd-n and even-n cases, we have the same limit expression

$$\lim_{\varepsilon\to 0} J_{z1} = -\sigma_2\sigma_{n-2}\bar{\phi}_\eta$$

with $\bar{\phi}_\eta$ some average value of $\phi(0,z)$ over the interval $0 \le z \le \eta$. In returning to (4.23),

$$\begin{aligned}\int d\tau^{(n)}\phi\,\Delta^{(n)}_{r,z}\Xi_n &= \lim_{\varepsilon\to 0}\left(\lim_{\eta\to 0} J_{r1}\right) = \lim_{\eta\to 0}\left(\lim_{\varepsilon\to 0} J_{z1}\right)\\ &= -\sigma_2\sigma_{n-2}\phi(0,0) = -(n-2)\sigma_n\phi(0,0), \end{aligned}\tag{4.28}$$

and our proof is finished. The appearance of the factor $(n-2)$ is due to that it was removed in the solution (4.14a). Thus, (4.28) is equivalent to equation (4.13a).

Below we shall use (4.28) in an equivalent form

$$\int_0^\infty dr\, r\int_0^\infty dz\, z^{n-3}\phi\,\Delta^{(n)}_{r,z}\Xi_n \equiv \left\langle rz^{n-3}\phi,\, \Delta^{(n)}_{r,z}\Xi_n\right\rangle = -\phi(0,0). \tag{4.29}$$

The recurrent way. Let us propose that we have proved, perhaps by the first method, that

$$\Delta^{(3)}_{r,z}\Xi_3(r,z) = -\frac{\delta_+(r)}{r}\,\delta_+(z)$$

and

$$\Delta^{(4)}_{r,z}\Xi_4(r,z) = -\frac{\delta_+(r)}{r}\,\frac{\delta_+(z)}{z}\,,$$

after that we can use the mathematical induction method. We prove that if

$$\Delta^{(k-2)}_{r,z}\Xi_{k-2}(r,z) = -\,\frac{\delta_+(r)}{r}\,\frac{\delta_+(z)}{z^{k-5}}\,, \tag{4.30}$$

then

$$\Delta^{(k)}_{r,z}\Xi_{k}(r,z) = -\,\frac{\delta_+(r)}{r}\,\frac{\delta_+(z)}{z^{k-3}}\,, \tag{4.31}$$

$k \geq 5$ being an integer number.

Equation (4.30) means that [see (4.29)]

$$\left\langle rz^{k-5}\phi,\ \Delta^{(k-2)}_{r,z}\Xi_{k-2}\right\rangle = -\phi(0,0) \tag{4.32}$$

for any $\phi(r,z) \ni \mathcal{D}(\mathcal{G})$. Here we recall that for every ϕ belonging to the mentioned class, first, any derivatives of ϕ belong to the same class, and, second, for any function $F(r,z)$ the following equality holds:

$$\left\langle \phi,\ \frac{\partial F}{\partial z}\right\rangle = -\left\langle F,\ \frac{\partial \phi}{\partial z}\right\rangle .$$

The required proof is given below by a chain of relations with using the transformation $f_L^{(k,k-2)}$:

$$\begin{aligned}\left\langle rz^{k-3}\phi,\ \Delta^{(k)}_{r,z}\Xi_k\right\rangle &= \left\langle rz^{k-3}\phi,\ f_L^{(k,k-2)}\Delta^{(k-2)}_{r,z}\Xi_{k-2}\right\rangle\\ &= \frac{1}{k-4}\left\langle r\,\frac{\partial}{\partial z}\,(\phi z^{k-4}),\ \Delta^{(k-2)}_{r,z}\Xi_{k-2}\right\rangle\\ &= \left\langle rz^{k-5}\left(\phi + \frac{z}{k-4}\,\frac{\partial\phi}{\partial z}\right),\ \Delta^{(k-2)}_{r,z}\Xi_{k-2}\right\rangle = -\phi(0,0).\end{aligned} \tag{4.33}$$

In the latter expression, the second term in parentheses vanishes when integrated with $\Delta^{(k-2)}_{r,z}\Xi_{k-2}$. This is due to the (formal) equality $z\delta_+(z) = 0$ after using (4.30), or this fact can be verified immediately. The first term there just gives (4.32). Thus, (4.31) is proved.

5 Anti-Laplacians: entirely unknown properties

5.1 Connections of anti-Laplacians with Laplacians

Until now we have already introduced the anti-r-Laplacian and anti-double-Laplacian operators. Here, we define the remaining possible anti-Laplacians and derive some relations connecting the anti-Laplacians with Laplacians.

The way by which (2.6) was obtained suggests that we propose a relation given below. Consider for simplicity the nD space in the Cartesian coordinate frame $x_1, x_2, \ldots, x_n$, let $x = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. We denote

$$\Delta^{(n)}_{Cart} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

the $\Delta^{(n)}$ operator in the Cartesian frame. For any function F depending only on x: $F = F(x)$, and for any coordinate x_k, the following relation holds:

$$\Delta^{(n)}_{Cart}\left(\frac{x_k}{x^n}\,F(x)\right) = \frac{x_k}{x^n}\,\overline{\Delta}^{(n)}_x F(x) \tag{5.1}$$

where the operator

$$\overline{\Delta}_x^{(n)} \equiv x^{n-1}\,\frac{\partial}{\partial x}\,\frac{1}{x^{n-1}}\,\frac{\partial}{\partial x} = \frac{\partial^2}{\partial x^2} - \frac{n-1}{x}\,\frac{\partial}{\partial x}$$

should be called anti-Laplacian. We see that this definition is suitable for the x frame. (The reader has already understood that the prefix "anti" is given because of the inverse position of "x" and "$1/x$" in the first expression and the opposite sign in the second expression as compared to the Laplacian).

In fact, Eq.(5.1) has a covariant meaning: it is independent of a coordinate frame. We reformulate it in the x frame. Let λ_1 be a senior angle and x_1 be a Cartesian axis associated with λ_1. (See Appendix A. We could also take an arbitrary axis x_k involving other angles, however this means a more notorious derivation.) Then, using (A.2)

$$\Delta^{(n)}\left(\frac{\cos\lambda_1}{x^{n-1}}\,F(x)\right) = \frac{\cos\lambda_1}{x^{n-1}}\,\overline{\Delta}_x^{(n)}F(x) \tag{5.2}$$

In the (r,z) frame, the nD anti-r-Laplacian can be defined by analogy with (2.4):

$$\overline{\Delta}_r^{(n)} = r\,\frac{\partial}{\partial r}\,\frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{z^{n-3}}\,\frac{\partial}{\partial z}\,z^{n-3}\,\frac{\partial}{\partial z}, \tag{5.3}$$

and we also define the operator

$$\overline{\Delta}_z^{(n)} \equiv \frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} + z^{n-3}\,\frac{\partial}{\partial z}\,\frac{1}{z^{n-3}}\,\frac{\partial}{\partial z}, \tag{5.4}$$

which following our "taxonomy" is natural to call anti-z-Laplacian and which plays a key role in our investigations. It accomplishes our collection of anti-Laplacians for the $2+(n-2)$ splitting of the nD space.

Given the (r,z) frame, the relation (5.2) can be written independently for the r and/or z spaces (cf. Eq.(2.6) in 3 dimensions). In n dimensions, for any $F(r,z)$, the analogue of (2.6) is

$$\Delta^{(n)}\left(\frac{\cos\varphi}{r}\,F(r,z)\right) = \frac{\cos\varphi}{r}\,\overline{\Delta}_r^{(n)}F(r,z), \tag{5.5}$$

plus there are the relations in the z space:

$$\Delta^{(n)}\left(\frac{\cos\theta_1}{z^{n-3}}\,F(r,z)\right) = \frac{\cos\theta_1}{z^{n-3}}\,\overline{\Delta}_z^{(n)}F(r,z) \tag{5.6}$$

and

$$\Delta^{(n)}\left(\frac{\cos\phi}{r}\,\frac{\cos\theta_1}{z^{n-3}}\,F(r,z)\right) = \frac{\cos\varphi}{r}\,\frac{\cos\theta_1}{z^{n-3}}\,\overline{\overline{\Delta}}^{(n)}F(r,z), \tag{5.7}$$

which can be verified with the expression (A.6).

Besides of the relations (5.5), (5.6) and (5.7), there exist some evident differential relations. Let us turn again to the x frame. Then, for any function $F(x)$,

$$x^{n-1}\,\frac{\partial}{\partial x}\,\Delta_x^{(n)}F(x) = \overline{\Delta}_x^{(n)}\left(x^{n-1}\,\frac{\partial}{\partial x}\,F(x)\right) \tag{5.8a}$$

and

$$\frac{1}{x^{n-1}}\,\frac{\partial}{\partial x}\,\overline{\Delta}_x^{(n)}F(x) = \Delta_x^{(n)}\left(\frac{1}{x^{n-1}}\,\frac{\partial}{\partial x}\,F(x)\right). \tag{5.8b}$$

The equalities (5.8) are in fact identities after using explicit forms of operators. Thus, we have demonstrated a principle of constructing such the relations.

In the (r, z) frame, a variety of relations similar to (5.8a) and (5.8b) arises, based on the fact of commutativity of the derivatives $\partial/\partial r$ and $\partial/\partial z$. For example,

$$r\,\frac{\partial}{\partial r}\,\Delta^{(n)}_{r,z}F(r,z) = \overline{\Delta}^{(n)}_{r}\left(r\,\frac{\partial}{\partial r}\,F(r,z)\right), \tag{5.9a}$$

$$z^{n-3}\,\frac{\partial}{\partial z}\,\Delta^{(n)}_{r,z}F(r,z) = \overline{\Delta}^{(n)}_{z}\left(z^{n-3}\,\frac{\partial}{\partial z}\,F(r,z)\right), \tag{5.9b}$$

$$z^{n-3}\,\frac{\partial}{\partial z}\,\overline{\Delta}^{(n)}_{r}F(r,z) = \overline{\overline{\Delta}}^{(n)}\left(z^{n-3}\,\frac{\partial}{\partial z}\,F(r,z)\right). \tag{5.9c}$$

for any $F(r, z)$. There are no necessity to bring all the remaining relations which correspond to (5.8a) and (5.8b). We shall use below one of them, namely

$$\frac{1}{z^{n-3}}\,\frac{\partial}{\partial z}\,\overline{\Delta}^{(n)}_{z}F(r,z) = \Delta^{(n)}_{r,z}\left(\frac{1}{z^{n-3}}\,\frac{\partial}{\partial z}\,F(r,z)\right). \tag{5.10}$$

Moreover, we can in principle combine the relations of the type (5.5)–(5.7) and those of the type (5.9) and (5.10).

5.2 Potential-like solutions of homogeneous equations

In keeping in mind that the point-like source is located at the point $r = 0, z = 0$, it turns out that nD solutions to homogeneous equations involving the operator (5.4), i.e., the anti-z-Laplace equations

$$\overline{\Delta}^{(n)}_{z}\Psi_n \equiv \left(\frac{1}{r}\,\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} + z^{n-3}\,\frac{\partial}{\partial z}\,\frac{1}{z^{n-3}}\,\frac{\partial}{\partial z}\right)\Psi_n = 0, \tag{5.11}$$

are the simplified versions of the solutions (3.11) and (3.15), (3.23). As before, we should consider separately the cases of odd and even dimensions.

In the odd-nD case, the function

$$\Psi_n = {\sum_{k=3}^{n}}'\, a_{k,n}\,\frac{z^{k-3}}{\bar{R}^{k-2}} \tag{5.12}$$

is the solution to (5.11) with $a_{k,n}$ given by (3.12) and (3.13), recall that $\bar{R} = \sqrt{r^2 + z^2}$. It is worth noting that the recurrence relation (3.12) for $a_{k,n}$ is a necessary and sufficient condition of the validity of (5.11), without any auxiliary construction like the integral identity (3.9).

In the even-nD case, the functions

$$\Psi_4 = \ln\frac{1}{\bar{R}}, \tag{5.13a}$$

and

$$\Psi_n = \ln\frac{1}{\bar{R}} + {\sum_{k=6}^{n}}''\,\frac{1}{k-4}\,\frac{z^{k-4}}{\bar{R}^{k-4}} \tag{5.13b}$$

for $n \geq 6$ are the solution to (5.11).

The solutions (5.13) have an interesting property:

$$\frac{\partial}{\partial z}\Psi_n = -\frac{z^{n-3}}{\bar{R}^{n-2}}\,, \tag{5.14}$$

which will be used for finding point-like sources in Sec.6.

The asymptotics of the solutions (5.12) and (5.13) entirely repeats those (3.11) and (3.22), (3.23), respectively, with ρ replaced by r. The absence of integrals over an angular variable makes the 3D or 2D behaviour of (5.12) and (5.13) quite clear.

5.3 Transformations from $(n-2)$D quantities to nD ones

For anti-Laplacians there also exist a transformation converting an $(n-2)$D solution to an nD one, mutual for odd-nD and even-nD cases. The transformation acts starting from $n \geq 5$:

$$\Psi_n = f_A^{(n,n-2)}\,\Psi_{n-2} = -\frac{z^{n-3}}{n-4}\frac{\partial}{\partial z}\left(\frac{1}{z^{n-4}}\,\Psi_{n-2}\right). \tag{5.15}$$

Using the differential rearrangement

$$z^{n-3}\,\frac{\partial}{\partial z}\,\frac{1}{z^{n-4}}\,\overline{\Delta}_z^{(n-2)} = \overline{\Delta}_z^{(n)}\,z^{n-3}\,\frac{\partial}{\partial z}\,\frac{1}{z^{n-4}}\,,$$

we also find

$$\overline{\Delta}_z^{(n)}\Psi_n = f_A^{(n,n-2)}\overline{\Delta}_z^{(n-2)}\Psi_{n-2}. \tag{5.16}$$

By analogy to (4.18), we introduce the anti-z-Laplace set, A_n:

$$A_n = \{\Psi_n,\ \overline{\Delta}_z^{(n)}\Psi_n,\ \bar{s}_z^{(n)}\}. \tag{5.17}$$

where $\bar{s}_z^{(n)}$ is a point-like source searched for, which gives rise to the solution Ψ_n:

$$\overline{\Delta}_z^{(n)}\Psi_n = \bar{s}_z^{(n)},$$

and whose explicit form we are just going to find in the next section using the transformation

$$\bar{s}_z^{(n)} = f_A^{(n,n-2)}\bar{s}_z^{(n-2)}. \tag{5.18}$$

This is in contrast to Sec.4, where the validity of the transformation $f_L^{(n,n-2)}$ acting on sources was merely checked. In compact notations, the equalities (5.15), (5.16) and (5.18) acquire the form

$$A_n = f_A^{(n,n-2)}A_{n-2}. \tag{5.19}$$

The equality (5.19) can also be resolved similarly to Sec.4: Any nD quantity from the anti-z-Laplace set (5.17) can be expressed via the 3D or 4D one. For odd n,

$$A_n = f_A^{(n,n-2)}f_A^{(n-2,n-4)}\dots f_A^{(5,3)}A_3 = \frac{z^{n-3}}{(n-4)!!}\left(-\frac{\partial}{\partial z}\,\frac{1}{z}\right)^{(n-3)/2}A_3, \tag{5.20}$$

and for even n,

$$A_n = f_A^{(n,n-2)}f_A^{(n-2,n-4)}\dots f_A^{(6,4)}A_4 = \frac{z^{n-3}}{(n-4)!!}\left(-\frac{\partial}{\partial z}\,\frac{1}{z}\right)^{(n-4)/2}\frac{A_4}{z}\,. \tag{5.21}$$

6 Obtaining the point sources for potential-like solutions to anti-z-Laplace equations

6.1 The simple odd-nD case

In the odd-nD case, the 3D quantities of the Laplace and anti-Laplace sets coincide: $A_3 = L_3$. Indeed, $\Delta^{(3)}_{r,z} = \overline{\Delta}^{(3)}_z$, $\Xi_3 = \Psi_3$ and $s^{(3)}_{r,z} = \bar{s}^{(3)}_z$. This fact outlines several ways to calculate $\bar{s}^{(n)}_z$. The first of them, the most simple and the most formal is to apply formula (5.20) to $\bar{s}^{(3)}_z$. This means that we in fact apply the transformation $f^{n,n-2}_A$ by required number of times to both the l.h.s. and r.h.s. of the 3D Laplace equation,

$$\Delta^{(3)}_{r,z} = s^{(3)}_{r,z},$$

and state that the transformed r.h.s. is a source we search for, the result being

$$\bar{s}^{(n)}_z = -\left(\sum_{k=3}^{n}{}' a_{k,n}\right)\frac{\delta_+(r)}{r}\,\delta_+(z) = -\frac{(n-3)!!}{(n-4)!!}\,\frac{\delta_+(r)}{r}\,\delta_+(z), \tag{6.1}$$

see (B.3). Perhaps, anybody could say that there should be taken more care in dealing with formal expressions. That is why we give the second (improved) way for obtaining the same result.

If to compare the expressions (4.20) and (5.20) for the transformations of the Laplace and anti-Laplace sets, respectively, it is clear that the later differs from the former by the presence of the factor z^{n-3} and by the permutation of $1/z$ and $\partial/\partial z$. This fact enables us to express algebraically any quantity A_n via $L_3, L_5, \ldots, L_{n-2}, L_n$. The expected expression,

$$A_n = \sum_{k=3}^{n}{}' a_{k,n} z^{k-3} L_k, \tag{6.2}$$

can be obtained by a recurrent way [cf. (5.12)]. Let us choose the second quantities from the sets (4.18) and (5.17), then from (6.2)

$$\overline{\Delta}^{(n)}_z \Psi_n = \sum_{k=3}^{n}{}' a_{k,n} z^{k-3} \Delta^{(k)}_{r,z}\Xi_k. \tag{6.3}$$

Further, we multiply (6.3) on $r\phi$, where $\phi(r,z)$ belongs to the above-mentioned class of trial functions, and perform integration with respect to r and z, cf. (4.29):

$$\left\langle r\phi,\, \overline{\Delta}^{(n)}_z \Psi_n\right\rangle = \sum_{k=3}^{n}{}' a_{k,n}\left\langle r z^{k-3}\phi,\, \Delta^{(k)}_{r,z}\Xi_k\right\rangle. \tag{6.4}$$

The k-th integrals on the r.h.s. of (6.4) were already calculated, see (4.33), thus

$$\left\langle r\phi,\, \overline{\Delta}^{(n)}_z \Psi_n\right\rangle = -\left(\sum_{k=3}^{n}{}' a_{k,n}\right)\phi(0,0), \tag{6.5}$$

meaning the validity of (6.1).

At least, the third and the most accurate way is to examine the limit of the (4.23)-type:

$$\left\langle r\phi,\, \overline{\Delta}^{(n)}_z \Psi_n\right\rangle = \lim_{\varepsilon,\eta\to 0}(\bar{J}_r + \bar{J}_z)$$

with the following surface terms obtained with the use of (6.3):

$$\bar{J}_r = \left[\int_0^{\varepsilon} dr\, r \left(\phi \sum_{k=3}^{n}{}' a_{k,n} z^{k-3} \frac{\partial \Xi_k}{\partial z} - \frac{\partial \phi}{\partial z}\, \Psi_n \right) \right]_{z=\eta},$$

$$\bar{J}_z = \left[\int_0^{\eta} dz\, r \left(\phi \frac{\partial \Psi_n}{\partial r} - \frac{\partial \phi}{\partial r}\, \Psi_n \right) \right]_{r=\varepsilon}.$$

The result, being independent of the order of taking limits with respect to ε and η, certainly coincides with (6.5).

In order to avoid a sum factor on the r.h.s. of (6.1) or (6.5), it is worth replacing the coefficients $a_{k,n}$ in (5.12) by the renormalized ones, $b_{k,n}$ [see Appendix B and especially (B.5)]. Thus, our final result is that for odd $n \geq 3$ the function

$$\bar{\Psi}_n = \sum_{k=3}^{n}{}' \frac{(k-4)!!\,(n-k-1)!!}{(k-3)!!\,(n-k)!!} \frac{z^{k-3}}{\bar{R}^{k-2}}$$

satisfies the equation

$$\overline{\Delta}_z^{(n)} \bar{\Psi}_n \equiv \left(\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + z^{n-3} \frac{\partial}{\partial z} \frac{1}{z^{n-3}} \frac{\partial}{\partial z} \right) \bar{\Psi}_n = -\, \frac{\delta_+(r)}{r}\, \delta_+(z).$$

Due to this renormalization, the n dependence in the source on the r.h.s. of the latter equation has disappeared.

6.2 The difficult even-nD case

For even n, the quantity A_n cannot be reduced to any combination of the quantities L_k. Certainly, $A_2 = L_2$ as a degenerate case, however our transformation starts to work with $A_4 \neq L_4$, cf. (4.21) and (5.21). Fortunately, the solutions (5.13) possess the property (5.14). Let us recall the relation (5.10) and impose $F(r,z) = \Psi_n$, then combining equations (5.10),(5.14) and (4.13) gives

$$\frac{1}{z^{n-3}} \frac{\partial}{\partial z}\, \overline{\Delta}_z^{(n)} \Psi_n = -\Delta_{r,z}^{(n)} \left(\frac{1}{\bar{R}^{n-2}} \right) = \frac{\delta_+(r)}{r} \frac{\delta_+(z)}{z^{n-3}}. \tag{6.6}$$

After multiplying (6.6) on z^{n-3} and taking the antiderivative with the use of the formal equality $\Theta_+'(z) = \delta_+(z)$,

$$\overline{\Delta}_z^{(n)} \Psi_n = \frac{\delta_+(r)}{r}\, \Theta_+(z) + \tilde{f}(r) \tag{6.7}$$

where $\tilde{f}(r)$ is unknown as yet generalized function, and $\Theta_+(z)$ is the Heaviside step function

$$\Theta_+(x) = \begin{cases} 0, & \text{for } x \leq 0; \\ 1, & \text{for } x > 0. \end{cases}$$

The equality (6.7) suggests us to explicitly calculate the case $n = 4$ with the Θ source, keeping in mind to further transform it to arbitrary even n.

Let us find the function $\tilde{\Psi}_4(r,z)$ which satisfies the equation

$$\overline{\Delta}_z^{(4)} \tilde{\Psi}_4 = \frac{\delta_+(r)}{r}\, \Theta_+(z). \tag{6.8}$$

Combining (6.8) with (5.6) for $n=4$ and with $F(r,z)$ replaced by $\tilde\Psi_4(r,z)$, and redenoting $\theta_1=\theta$ lead to the equation

$$\Delta^{(4)}_{r,z,\theta}\left(\frac{\cos\theta}{z}\,\tilde\Psi_4(r,z)\right)=\frac{\cos\theta}{z}\,\overline{\Delta}^{(4)}_z\tilde\Psi_4(r,z)=\frac{\cos\theta}{z}\,\frac{\delta_+(r)}{r}\,\Theta_+(z)\equiv-\sigma_4\mu(r,z,\theta). \quad (6.9)$$

The fundamental solution to the Poisson equation (4.1) in the case $n=4$ is

$$\Phi_4=\frac{1}{2|\vec R-\vec R'|^2}$$

where

$$|\vec R-\vec R'|^2=r^2+r'^2-2r'r\cos(\varphi-\varphi')+z^2+z'^2-2z'z\cos(\theta-\theta').$$

Thus, similarly to a derivation done in Sec.2 and in accordance with a standard rule, the solution to (6.8) can be written as follows

$$\frac{\cos\theta}{z}\,\tilde\Psi_4=\frac{1}{2}\int_{\tau^{(4)\prime}}d\tau^{(4)\prime}\,\frac{\mu(r',z',\theta')}{|\vec R-\vec R'|^2}=-\frac{1}{2\sigma_4}\int\limits_0^\infty dr'\int\limits_0^{2\pi}d\varphi'\int\limits_0^\infty dz'\int\limits_0^{2\pi}d\theta'\times$$

$$\frac{\cos\theta'\delta_+(r')\Theta_+(z')}{r^2+r'^2-2r'r\cos(\varphi-\varphi')+z^2+z'^2-2z'z\cos(\theta-\theta')}\,. \quad (6.10)$$

Strictly speaking, the standard rule prescribes to deal with compact sources, although the source in (6.9) is not compact. Nevertheless, the integral (6.10) exists. For accuracy, we could make a limit procedure originating from integration over a confined space domain, and then coming to the whole space.

Now we shall perform over (6.10) the three subsequent operations: 1) integration with respect to φ' with the use of (C.4), 2) integration with respect to r', and 3) the procedure of removing the factor $\cos\theta$ after imposing $\theta'=\theta+\beta$ which was already described in Sec.2. This leads to

$$\tilde\Psi_4=-\frac{1}{\pi}\,z\int\limits_0^\infty dz'\int\limits_0^\pi\frac{d\beta\,\cos\beta}{r^2+z^2+z'^2-2z'z\cos\beta}\,. \quad (6.11)$$

The next step is the integration of (6.11) with respect to β using (C.5):

$$\tilde\Psi_4=-\frac{1}{2}\int\limits_0^\infty\frac{dz'}{z'}\left(\frac{r^2+z^2+z'^2}{[(r^2+z^2)^2+2(r^2-z^2)z'^2+z'^4]^{1/2}}-1\right). \quad (6.12)$$

The integral (6.12) can be calculated with the use of (C.6). This final integration gives

$$\tilde\Psi_4=\ln\frac{1}{\sqrt{r^2+z^2}}-\ln\frac{1}{r}\,.$$

Now, we have obtained the solution corresponding to the Θ source (6.8). It differs from the announced solution to the homogeneous equation by the only last term. However, the later is the degenerate 2D solution (4.14b) taken with an opposite sign, and that is why it is simultaneously the z independent solution to the equation [cf. (4.13b)]:

$$\overline{\Delta}^{(4)}_z\Xi_2=\Delta^{(2)}\Xi_2=\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}\,\Xi_2=-\,\frac{\delta_+(r)}{r}\,.$$

Taking the sum of the solutions $\tilde{\Psi}_4$ and Ξ_2, we obtain that the function (5.13a),

$$\Psi_4 = \tilde{\Psi}_4 + \Xi_2 = \ln \frac{1}{\bar{R}}\,, \tag{6.13}$$

satisfies the equation with a point-like source

$$\overline{\Delta}_z^{(4)} \Psi_4 = -\,\frac{\delta_+(r)}{r}\,[1 - \Theta_+(z)] \equiv \bar{s}_z^{(4)}. \tag{6.14}$$

Indeed, for the range of values $z \geq 0$,

$$1 - \Theta_+(z) = \begin{cases} 1, & \text{for } z = 0; \\ 0, & \text{for } z > 0. \end{cases} \tag{6.15}$$

We should remark that the situation is somewhat paradoxical. The function (6.15) is finite (nonsingular) in the point $z = 0$; and the source in (6.14) although located at the point $r = 0, z = 0$ is more "weak" than the true δ source. However, the logarithmic divergency in (6.13) when reaching zero is more weak than the divergency $\sim \bar{R}^{-2}$ as it could be in the case $n = 4$ for the true δ source. If we tried to extract this source by the standard method involving trial functions as it was done before, we would obtain zero when integrating over the whole 4D space, although the derivative of (6.15) with respect to z is the $-\delta_+$ function as usual.

There exists a theorem stating that every generalized function concentrated at a point is a combination of the δ functions and their derivatives [8], the proof being done in terms of finite functionals on continuous functions. The theorem has a consequence that a solution to the Laplace equation with a power-law singularity is generated by the above combination. It is also extended onto partial differential equations with constant coefficients (in a certain frame, if any). Our situation is more sophisticated. The author's opinion is that, first, there are no finite nonzero functional determined by the generalized function (6.15) in any class of trial (continuous) functions. Second, the equations with anti-Laplacians for every $n > 3$ and the even-nD solutions considered are not the equations and solutions of the above type. Certainly, additional investigations to this situation from the viewpoint of functional analysis are required.

In order to extend the obtained 4D source on subsequent even n, we can now use the transformation $f_A^{(n,n-2)}$. One could easily ensure that it does not change the form of the source,

$$f_A^{(n,n-2)}\,[1 - \Theta_+(z)] = -\,\frac{z^{n-3}}{n-4}\,\frac{\partial}{\partial z}\left(\frac{1}{z^{n-4}}\,[1 - \Theta_+(z)]\right) = 1 - \Theta_+(z),$$

due to the formal equality $z\delta_+(z) = 0$. Our previous remark can be transferred mutatis mutandis to the nD case as well.

Thus, we establish that for even $n \geq 6$ the function (5.13b),

$$\Psi_n = \ln \frac{1}{\bar{R}} + \sum_{k=6}^{n}{}'' \,\frac{1}{k-4}\,\frac{z^{k-4}}{\bar{R}^{k-4}}\,,$$

is the solution to the equation

$$\overline{\Delta}_z^{(n)} \Psi_n = -\,\frac{\delta_+(r)}{r}\,[1 - \Theta_+(z)] \equiv \bar{s}_z^{(n)}.$$

Fortunately, we have chose from the beginning such a mutual coefficient at Ψ_n that $\bar{s}_z^{(n)}$ has no dependence on n.

7 Returning to the vortex-ring-like solutions. Conclusion

As before, introduce the anti-double-Laplacian set, D_n,

$$D_n = \{\psi_n,\ \overline{\overline{\Delta}}^{(n)}\psi_n,\ \overline{\overline{s}}^{(n)}\}.$$

Given such a powerful tool as the transformation $f_A^{(n,n-2)}$ in (5.15), we have no problems in constructing any quantity D_n from D_{n-2}. Indeed, the distinction between corresponding A_n and D_n is entirely referred to their "r parts", the "z parts" being unchanged. Even without such an explanation, it can be immediately verified that

$$D_n = f_A^{(n,n-2)} D_{n-2}.$$

Expected results can be formulated as before separately for the two cases.

In the odd-nD case, we should start with D_3: As to the operators, $\overline{\overline{\Delta}}^{(3)}$ coincides with the operator (2.4) which should be redenoted as $\overline{\Delta}_r^{(3)}$: $\overline{\overline{\Delta}}^{(3)} = \overline{\Delta}_r^{(3)}$. Unlike Sec.2, where we have defined Ω_φ by a traditional way, here we make another definition:

$$\int_0^\infty dr\, r \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dz\ \Omega_\varphi = 4\pi, \tag{7.1}$$

in order for the source $\overline{\overline{s}}^{(3)}$ to have the form

$$\overline{\overline{s}}^{(3)} = -\delta(r-a)\delta_+(z) \tag{7.2}$$

instead of (2.7). [We have imposed $\delta(z) = \delta(\vec{z}) = \delta_+(z)/2$ in (7.2) according to the general formula (4.3).] Hence, the 3D vortex-ring solution can be rewritten in the form

$$\psi_3 = \frac{1}{\pi}\, r \int_0^\pi d\alpha \cos\alpha\, \frac{1}{R}\,,$$

instead of (2.11), where, recalling from (3.7),

$$R = [r^2 + a^2 - 2ar\cos\alpha + z^2]^{1/2}.$$

The final result is that, for odd $n \geq 3$, the function

$$\psi_n = \frac{1}{\pi}\, r \sum_{k=3}^{n}{}' \frac{(k-4)!!\,(n-k-1)!!}{(k-3)!!\,(n-k)!!} \int_0^\pi d\alpha \cos\alpha\, \frac{z^{k-3}}{R^{k-2}}$$

satisfies the equation

$$\overline{\overline{\Delta}}^{(n)}\psi_n \equiv \left(r\,\frac{\partial}{\partial r}\,\frac{1}{r}\,\frac{\partial}{\partial r} + z^{n-3}\,\frac{\partial}{\partial z}\,\frac{1}{z^{n-3}}\,\frac{\partial}{\partial z} \right)\psi_n = -\delta(r-a)\,\delta_+(z) \equiv \overline{\overline{s}}^{(n)}.$$

In the even-nD case, the same problem as that in the subsection 6.2 arises, however, it can be resolved by one-to-one correspondence at each step of a derivation. We only recall that instead of the solution (4.13b) to equation (4.14b), we have to use the solution

(3.25) to equation (3.24) with a source $\propto \delta(r-a)$. Thus, for $n=4$ and even $n \geq 6$, the functions

$$\psi_4 = \frac{1}{\pi}\, r \int\limits_0^\pi d\alpha \cos\alpha \,\ln\frac{1}{R} \tag{7.3}$$

and

$$\psi_n = \frac{1}{\pi}\, r \int\limits_0^\pi d\alpha \cos\alpha \,\ln\frac{1}{R} \; + \frac{1}{\pi}\, r \sum_{k=6}^{n}{}'' \frac{1}{k-4} \int\limits_0^\pi d\alpha \cos\alpha \,\frac{z^{k-4}}{R^{k-4}}\,,$$

respectively, are the solutions to the equation

$$\overline{\overline{\Delta}}^{(n)} \psi_n = -\delta(r-a)\,[1-\Theta_+(z)] \equiv \overline{\overline{s}}^{(n)}.$$

Now, several concluding remarks are in order.

1. Certainly, every nD solution to equations involving an anti-z-Laplacian or anti-double-Laplacian can in principle be obtained by the procedure described in Sec.2, namely, by using the relations (5.6) and (5.7), and by integrating a fundamental solution to the Poisson equation multiplied by a corresponding source. However, such the way gives rise to great practical difficulties. In the (r,z) frame, an immediate integration is still relatively easy for $n=4$, as it was demonstrated in the subsection 6.2, but even for $n=5$ an integrand arises which contains special functions with arguments involving trigonometric functions. These difficulties grows when coming to each subsequent n. That is why the transformation $f_A^{(n,n-2)}$ saves the situation: There are no problems in obtaining solutions with an arbitrary (large) n.

2. All the obtained here solutions (or their integrands) have one more advantage that they are some algebraic functions of z and $\bar{R}$ (or R) plus the logarithmic function of $\bar{R}$ (or R) for even n. In the latter case both the anti-z-Laplace and anti-double-Laplace solutions corresponding to sources with $\delta_+(z)$ instead of $1-\Theta_+(z)$ are also feasible. However, they involve inverse trigonometric functions of r and z, and have no such remarkable announced properties. Note that by virtue of the relations (5.9b) and (5.9c), the odd-n solutions related to a $\delta'_+(z)$ are also feasible, but this subject is outside the framework of our study.

3. Is was technically more simple to work with the anti-z-Laplacians than with the anti-double-Laplacians. It was also more simple to extract the form of point sources than that of ring sources. Moreover, this study has clarified the fact that the solutions corresponding to both the operators are of a similar type – it does not matter whether their sources are ring-like or point-like ones.

However, there also exist ring-like solutions corresponding to the anti-z-Laplacians. We add them to the list of previous solutions. They are "scalar" in their "r parts" without changes in the "z parts", so that the transformation $f_A^{(n,n-2)}$ is working as before. As the result, for odd n, the function

$$\chi_n = \frac{1}{\pi} \sum_{k=3}^{n}{}' \frac{(k-4)!!\,(n-k-1)!!}{(k-3)!!\,(n-k)!!} \int\limits_0^\pi d\alpha\, \frac{z^{k-3}}{R^{k-2}} \tag{7.4}$$

$(n \geq 3)$ satisfies the equation

$$\overline{\Delta}_z^{(n)} \chi_n = -\frac{\delta(r-a)}{r}\, \delta_+(z).$$

While for $n = 4$ and even $n \geq 6$, the functions

$$\chi_4 = \frac{1}{\pi} \int\limits_0^\pi d\alpha \, \ln \frac{1}{R} \tag{7.5}$$

and

$$\chi_n = \frac{1}{\pi} \int\limits_0^\pi d\alpha \, \ln \frac{1}{R} + \frac{1}{\pi} \sum_{k=6}^{n}{}'' \frac{1}{k-4} \int\limits_0^\pi d\alpha \, \frac{z^{k-4}}{R^{k-4}} \,, \tag{7.6}$$

respectively, are the solutions to the equation

$$\overline{\Delta}_z^{(n)} \chi_n = -\frac{\delta(r-a)}{r} \, [1 - \Theta_+(z)].$$

In both the cases, the integral identity

$$\int\limits_0^\pi d\alpha \left(\frac{\cos\alpha}{R^k} - k \, \frac{ar \sin^2\alpha}{R^{k+2}} \right) = \int\limits_0^\pi d\alpha \, \frac{\partial}{\partial\alpha} \left(\frac{\cos\alpha}{R^k} \right) = 0$$

is necessary for immediate verifying that the functions (7.4), (7.5) and (7.6) satisfy the homogeneous equations $\overline{\Delta}_z^{(n)} \chi_n = 0$.

4. Until now we have said nothing about solutions corresponding to the $(n > 3)$-dimensional anti-r-Laplacians (5.3). The extension of equation (2.4) onto the nD space is

$$\overline{\Delta}_r^{(n)} \xi_n \equiv \left(r \, \frac{\partial}{\partial r} \, \frac{1}{r} \, \frac{\partial}{\partial r} + \frac{1}{z^{n-3}} \, \frac{\partial}{\partial z} \, z^{n-3} \, \frac{\partial}{\partial z} \right) \xi_n = -r \Omega_\varphi^{(n)}(r, z) \equiv \bar{s}_r^{(n)}. \tag{7.7}$$

We define $\Omega_\varphi^{(n)}$ by a way which generalizes (7.1) to the nD case:

$$\int_{\tau^{(n)}} d\tau^{(n)} \, \Omega_\varphi^{(n)} = (n-2)\sigma_n \tag{7.8}$$

with $d\tau^{(n)}$ determined by (A.4). (Here, both the odd-nD and even-nD cases can be considered in common.) After that the source in (7.7) acquires the form

$$\bar{s}_r^{(n)} = -\sigma_{n-2} \delta(r-a) \, \delta(\vec{z}) = -\delta(r-a) \, \frac{\delta_+(z)}{z^{n-3}}.$$

This case is very simple. It can be resolved using the relation (5.5) and dealing with the Cartesian frame in the z space where $\delta(\vec{z}) = \delta(z_1)\delta(z_2)\ldots\delta(z_{n-2})$. The answer is that the function

$$\xi_n = \frac{1}{\pi} \, r \int\limits_0^\pi d\alpha \cos\alpha \, \frac{1}{R^{n-2}}$$

for all $n \geq 3$ is the solution to the equation

$$\overline{\Delta}_r^{(n)} \xi_n = -\delta(r-a) \, \frac{\delta_+(z)}{z^{n-3}} \, .$$

We see that this solution has no attractive properties of those considered before. It exhibits a typical nD-space behaviour. Nevertheless, due to the choice of (7.8), the anti-r-Laplacian set, B_n:

$$B_n = \{\xi_n, \, \overline{\Delta}_r^{(n)} \xi_n, \, \bar{s}_r^{(n)}\},$$

also has the property of the type (4.19):

$$B_n = f_L^{(n,n-2)} B_{n-2}$$

where the transformation $f_L^{(n,n-2)}$ suitable for the Laplace set is involved.

5. Emphasize one more that coming from a "Laplacian part" of an operator to an "anti-Laplacian part" for a given subspace means coming from scalars to other geometrical objects. Now it is already clear, by analogy with Sec.2, that in the case $n = 4$ the presented solutions, i.e. the functions (6.13) and (7.3), are the θ components of vectors in the 2-dimensional z space orthogonal to the radial direction. For larger n, the situation seems to be more intricate because components of some polyvector are dealth with. The author is going to devote to this subject further studies.

Appendix B. More about $a_{k,n}$

The coefficients $a_{k,n}$ can be represented in the form equivalent to (3.13):

$$a_{k,n} = \frac{(n-3)!!}{(n-4)!!} \frac{(k-4)!!\,(n-k-1)!!}{(k-3)!!\,(n-k)!!}\,. \tag{B.1}$$

Note that $(-1)!! = 1$ and $0!! = 1$ by definition. It is clear from (B.1), that these coefficients possess the symmetry property

$$a_{k,n} = a_{n-k+3,n}. \tag{B.2}$$

In particular,

$$a_{n,n} = a_{3,n} = 1.$$

Besides the recurrence relation (3.12), there exists one more recurrence relation for $k \leq n-2$:

$$a_{k,n} = a_{k,n-2} \frac{(n-3)(n-k-1)}{(n-4)(n-k)}\,.$$

This formula permits one to rapidly calculate the triangle of the numerical values of $a_{k,n}$. We give it below for $n = 3, 5, 7, 9, 11, 13, 15$.

$n=3$	1						
$n=5$	1	1					
$n=7$	1	2/3	1				
$n=9$	1	3/5	3/5	1			
$n=11$	1	4/7	18/35	4/7	1		
$n=13$	1	5/9	10/21	10/21	5/9	1	
$n=15$	1	6/11	5/11	100/231	5/11	6/11	1
...				...			

Let us prove that

$$\sum_{k=3}^{n}{}' \, a_{k,n} = \frac{(n-3)!!}{(n-4)!!}\,. \tag{B.3}$$

To do this, we renormalize the coefficients $a_{k,n}$:

$$a_{k,n} = \frac{(n-3)!!}{(n-4)!!}\, b_{k,n}, \tag{B.4}$$

$$b_{k,n} = \frac{(k-4)!!\,(n-k-1)!!}{(k-3)!!\,(n-k)!!}\,. \tag{B.5}$$

According to (B.3) and (B.4), we now must prove that

$$\sum_{k=3}^{n}{}' \; b_{k,n} = 1.$$

Let us impose $k = 2l + 1$ and $n = 2q + 1$ and use the equality

$$\binom{-1/2}{m} = (-1)^m \,\frac{(2m-1)!!}{(2m)!!}\,,$$

see [9], p. 772, where by definition

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}\,.$$

The coefficients (B.5) can be now expressed as follows

$$b_{k,n} = \tilde{b}_{l,q} = (-1)^{q-1}\binom{-1/2}{l-1}\binom{-1/2}{q-l}.$$

Let us write the expression for the following sum

$$\sum_{l=1}^{q}\binom{-1/2}{l-1}\binom{-1/2+\epsilon}{q-l} = \binom{-1+\epsilon}{q-1} \tag{B.6}$$

in accordance with the handbook equality for any complex a and b (see loc. cit., no. 13 on p. 616):

$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}.$$

It is necessary to obtain the sum (B.6) for $\epsilon = 0$. However, in this case (B.6) contains an indeterminate form on its r.h.s. In order to evaluate it we take the limit $\epsilon \to 0$:

$$\begin{aligned}\lim_{\epsilon\to 0}\binom{-1+\epsilon}{q-1} &= \lim_{\epsilon\to 0}\frac{\Gamma(\epsilon)}{\Gamma(q)\Gamma(1-q+\epsilon)}\\ &= \lim_{\epsilon\to 0}\frac{\Gamma(\epsilon)}{\Gamma(q-\epsilon)\Gamma(1-q+\epsilon)} = \lim_{\epsilon\to 0}\frac{\sin\pi(q-\epsilon)}{\epsilon\,\pi} = (-1)^{q-1}\end{aligned}$$

where we have replaced the finite value $\Gamma(q)$ on $\Gamma(q-\epsilon)$ under the sign of limit. At least,

$$\sum_{k=3}^{n}{}' \; b_{k,n} = \sum_{l=1}^{q}\tilde{b}_{l,q} = (-1)^{q-1}\lim_{\epsilon\to 0}\binom{-1+\epsilon}{q-1} = 1,$$

that finishes our proof. Perhaps, a more elegant proof is available, however, the author could not find it.

We also give the triangle of the coefficients $b_{k,n}$ (similar to that of $a_{k,n}$) because our final expressions for odd-nD solutions contain just $b_{k,n}$.

$n=3$	1						
$n=5$	1/2	1/2					
$n=7$	3/8	1/4	3/8				
$n=9$	5/16	3/16	3/16	5/16			
$n=11$	35/128	5/32	9/64	5/32	35/128		
$n=13$	63/256	35/256	15/128	15/128	35/256	63/256	
$n=15$	231/1024	63/512	105/1024	25/256	105/1024	63/512	231/1024
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It is necessary to obtain the sum (B.6) for $\epsilon = 0$. However, in this case (B.6) contains an indeterminate form on its r.h.s. In order to evaluate it we take the limit $\epsilon \to 0$:

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where we have replaced the finite value $\Gamma(q)$ on $\Gamma(q-\epsilon)$ under the sign of limit. At least,

$$\sum_{k=3}^{n}{}' b_{k,n} = \sum_{l=1}^{q}\tilde{b}_{l,q} = (-1)^{q-1}\lim_{\epsilon\to 0}\binom{-1+\epsilon}{q-1} = 1,$$

that finishes our proof. Perhaps, a more elegant proof is available, however, the author could not find it.

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$n=9$	5/16	3/16	3/16	5/16			
$n=11$	35/128	5/32	9/64	5/32	35/128		
$n=13$	63/256	35/256	15/128	15/128	35/256	63/256	
$n=15$	231/1024	63/512	105/1024	25/256	105/1024	63/512	231/1024
...				...			

Appendix C. The list of integrals

We give several integrals and a sum formula used in our derivation following [9].

1). p. 91, no. 7 ($n = 2q + 1$):

$$\int dz \, \frac{z^{n-3}}{(\varepsilon^2 + z^2)^{n/2}} =$$

$$\int dz \frac{z^{2(q-1)}}{(\varepsilon^2 + z^2)^{q+1/2}} = \frac{z^{2q-1}}{(2q-1)\varepsilon^2(\varepsilon^2 + z^2)^{q-1/2}} \, , \tag{C.1}$$

2). p. 30, no. 6 [$n = 2(m + 2)$, $m = 0$ corresponding to $n = 4$]:

$$\int dz \, \frac{z^{n-3}}{(\varepsilon^2 + z^2)^{n/2}} =$$

$$\frac{1}{2} \int du \, \frac{u^m}{(\varepsilon^2 + u)^{m+2}} = -\frac{1}{2} \sum_{k=0}^{m} \binom{m}{k} \frac{(-\varepsilon^2)^{m-k}}{(m-k+1)(\varepsilon^2 + u)^{m-k+1}} \tag{C.2}$$

where we have denoted $u = z^2$ and

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}$$

is the standard definition for a binomial coefficient (integer m and k).

3). In the sum expression

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k}}{m-k+1} =$$

$$\frac{(-1)^m}{m+1} \left[\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} - (-1)^{m+1} \right] = \frac{1}{m+1} = \frac{2}{n-2}, \tag{C.3}$$

the first term (sum) in the square brackets is zero, see p. 606, no. 3.

4). p. 181, no. 5:

$$\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} \, , \tag{C.4}$$

5). p. 414, no. 22:

$$\int_0^{\pi} \frac{dx \, \cos x}{a + b\cos x} = \frac{\pi}{b} \left(1 - \frac{a}{\sqrt{a^2 - b^2}} \right) , \tag{C.5}$$

6). p. 102, no. 8:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c)}} = \frac{1}{\sqrt{a}} \, \ln \left| \frac{2ax + b}{2\sqrt{a}} + \sqrt{ax^2 + bx + c)} \right| . \tag{C.6}$$

References

[1] H. Lamb, *Hydrodynamics* (Cambridge University press, Cambridge, 1895).

[2] P. Appell, *Traité de mecanique rationelle*, v.3, *Equilibre et mouvement des milieux continus* (Gauthier-Villars, Paris, 1928).

[3] M.A. Lavrent'yev and B.V. Shabat, *Problems of Hydrodynamics and Their Mathematical Models* (Nauka, Moscow, 1977, in Russian).

[4] L.M. Milne-Thomson, *Theoretical Hydrodynamics* (London–New-York, Macmillan and Co. LTD, 1960).

[5] J.A. Schouten, *Tensor Analysis for Physicists* (The Clarendon Press, Oxford, 1951).

[6] G.A. Korn and T.M. Korn, *Mathematical Handbook for Physicists and Engineers* (McCraw-Hill Book Company, New-York–London, 1961).

[7] I.M. Gel'fand and G.E. Shilov, *Generalized Functions and Operations with Them* (State Publishing House of Physical and Mathematical Literature, Moscow, 1958, in Russian).

[8] I.M. Gel'fand and G.E. Shilov, *The Spaces of Trial and Generalized Functions* (State Publishing House of Physical and Mathematical Literature, Moscow, 1958, in Russian).

[9] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series* (Nauka, Moscow, 1981, in Russian).